

An EDHEC-Risk Institute Publication

The Valuation of Privately-Held Infrastructure Equity Investments

Theoretical Framework and Data Collection Requirements

January 2015



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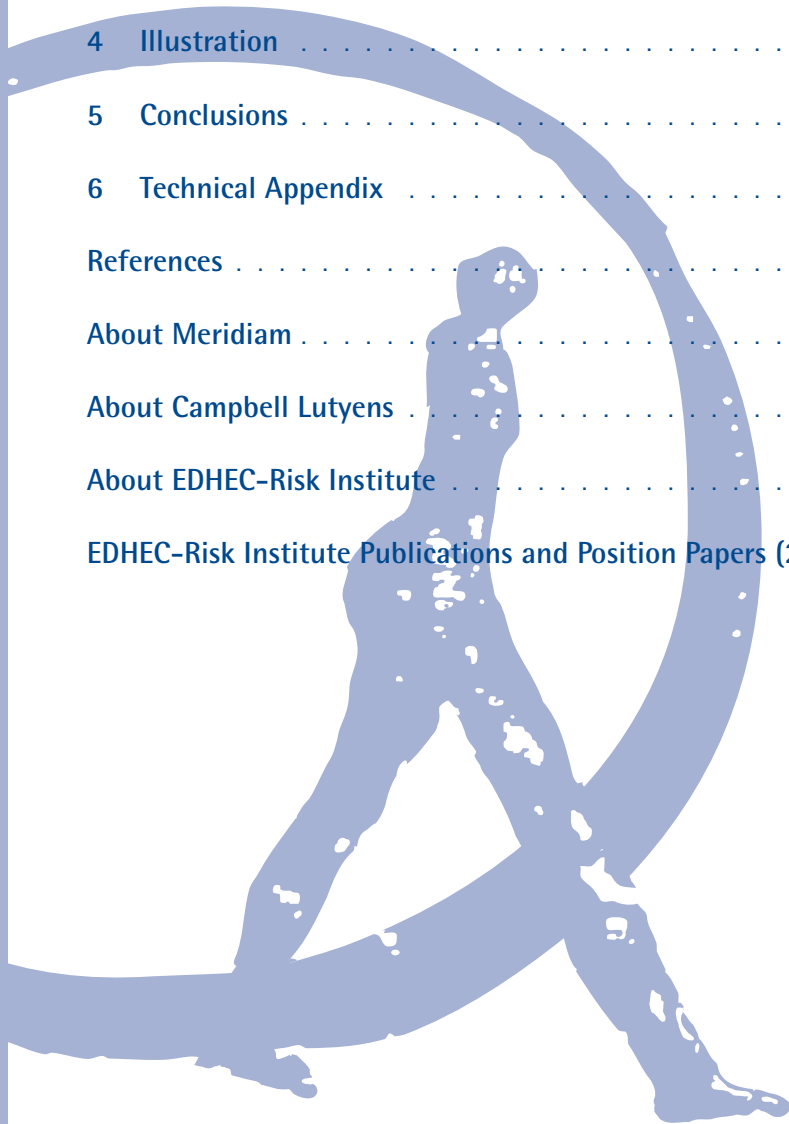
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Foreword

This paper, "The Valuation of Privately-Held Infrastructure Equity Investments", which is drawn from the Meridiam and Campbell Lutyens Research Chair at EDHEC on *Infrastructure Equity Investment Management and Benchmarking*, proposes a valuation framework for privately-held and very illiquid assets such as equity stakes in infrastructure projects.

Such a framework is one of the key steps identified by EDHEC-Risk Institute as part of a roadmap to design long-term infrastructure investment benchmarks that can take into account the nature of such assets as well as the paucity of available data.

Indeed, The design of an academically validated valuation framework, while necessary to ensure adequate performance measures, is constrained by the practical limitations of collecting private information that is scattered amongst many investors and is often confidential in nature.

The approach taken by the authors aims to balance the objective of using academically sound pricing models with that of requiring a parsimonious data input, thus making the necessary data collection process cost-efficient and realistic.


To address these issues, this paper develops a cash flow forecasting model and a pricing model that make use of powerful but simple Bayesian statistical principles, thus allowing the leveraging of available information as well as built-in learning, as and when new data become available.

This research also leads to the creation of a data collection template for infrastructure investors and their managers, which could be a useful starting point for a reporting standard of private infrastructure investment data and performance. With such a standard, industry-wide data collection can take place and the knowledge of the risk-adjusted performance of infrastructure equity investments can be improved to the point where asset allocation decisions and the calibration of prudential frameworks do not have to treat infrastructure investment as a known unknown anymore.

The next stage in the development of long-term investment benchmarks in infrastructure is the active collection of the required data from long-term investors in infrastructure on an ongoing basis, which EDHEC Business School will be pursuing in the coming years, with its industry partners.

We are grateful to Meridiam and Campbell Lutyens for their support of this study.

We wish you a thought-provoking, useful and informative read.



Frédéric Ducoulombier

Director of EDHEC Risk Institute-Asia

About the Authors



Frédéric Blanc-Brude is Research Director at EDHEC Risk Institute-Asia and heads the Institute's thematic research programme on infrastructure financing and investment. Prior to joining EDHEC-Risk Institute, he worked for ten years in the infrastructure sector and has been actively involved in transactions representing a cumulative value of more than USD6bn in Europe, Asia and the Middle East. Between 2008 and 2011, he headed the operations of a boutique consultancy specialised in energy and water projects in China. He has published his research widely and talks regularly presents at academic and industry conferences. He has taught finance at King's College London and designed and delivered executive seminars on infrastructure finance. He holds a PhD in Finance from the University of London.



Majid Hasan is Research Assistant at EDHEC Risk Institute-Asia and PhD Candidate at EDHEC Business School. His research interests include asset pricing in imperfect markets, and designing solutions to channel private capital towards public infrastructure. He holds a Masters in Applied Mathematics from Western University, Canada, and a Bachelors in Physics from the University of Punjab, Pakistan. Before joining EDHEC, he worked as a Research and Teaching Assistant at Western University, and the University of British Columbia, Canada.

Executive Summary



Executive Summary

This paper contributes a **rigorous valuation framework** to the debate on the benchmarking of privately-held infrastructure equity investments.

We define infrastructure equity investments as privately-held shareholdings in firms created to build, operate and maintain certain infrastructure projects or networks. While such firms are often created specifically for the purpose of developing and operating a given infrastructure project for a finite time period, private infrastructure equity may also correspond to open-ended stakes in firms that own and operate multiple infrastructure assets, such as utilities.

Our proposed approach is rooted in modern asset pricing and statistical inference theory, but remains a fully operational solution to the formation of performance expectations for sophisticated investors.

We also propose a parsimonious data collection template, which can be used on an industry-wide basis to improve existing knowledge of the performance of privately-held infrastructure equity investments on an ongoing basis.

Objectives

We aim to achieve the following objectives:

1. Determine the most appropriate **valuation framework** for privately-held equity investments in infrastructure projects or entities;

2. Design a **methodology** that can be readily applied given the current state of empirical knowledge and, going forward, at a minimum cost in terms of data collection;
3. Derive some of the most relevant valuation and performance **measures** for long-term equity investors and regulators;
4. Define a parsimonious data collection **template** that nevertheless allows meeting the three points above.

Challenges

The valuation of unlisted infrastructure project equity stakes requires addressing three significant challenges:

- a) **Endemic data paucity:** while primary and secondary market prices can be observed, sufficiently large and periodic samples, representative of different types of infrastructure projects at each point in their multi-decade lifecycle are unlikely to be available in each reporting period.
- b) **The term structure of expected returns:** the nature of such investments requires estimating a term structure of discount factors at different points in their lives that reflects the change in their risk profile. Indeed, *in expectation*, infrastructure investments can exhibit a dynamic risk profile determined by the sequential resolution of uncertainty, the frequent de-leveraging of the project company's balance sheet or the existence of a fixed-term to the investment which creates a time-varying duration.

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c) **The absence of a unique price** for a given investment in unlisted infrastructure, which springs from the fact that there is no traded equivalent to the payoff of infrastructure project equity. It follows that prices are partly driven by investor preferences and that substantial bid/ask spreads are likely.

The first point is partly a mundane aspect of the difficulties encountered when collecting data on private investments, but also a reflection of the nature of long-term equity investment in infrastructure. Indeed, the type of infrastructure projects that have been financed in the past are not necessarily representative of investment opportunities today. Thus, even if year-23 dividends for projects that were financed 24 years ago can be observed today, they may not be good predictors of year-23 dividends of projects financed 3 years ago. For example, projects financed in the early 1990s may have been in sectors where fewer projects exist today (e.g. telecoms) or rely on contractual structures or technologies that are not relevant to long-term investors in infrastructure today (e.g. coal-fired merchant power).

If data paucity is an endemic dimension of the valuation of privately-held infrastructure equity investment, i.e. *we must start from the premise that we cannot observe enough data to simply derive prices empirically*. Instead, we acknowledge a position of relative ignorance and aim to build into our approach the possibility of improving our knowledge as new observations that can be used to update

models of dividend distributions become available.

The second point about the term structure of expected returns has long been made in the finance literature: using such constant and deterministic discount rates is defective if projects have multiple phases and project risk changes over time as real-options are exercised by asset owners.

Indeed, a constant risk premium does not measure risk properly on a period by period basis, but rather implies that cash flows occurring further in the future are riskier than cash flows occurring earlier, which may not be the case, especially given the kind of sequential resolution of uncertainty which characterises infrastructure projects.

Using constant discount rates amounts to assuming that the risk-free rate, asset beta, and market risk premium are all deterministic and constant at all future points in time, while these variables are effectively time-varying and stochastic (that is, conditional on current information, future expected discount rates are stochastic).

In any case, the internal rate of return (IRR) of individual investments cannot be easily used to estimate performance at the **portfolio** level, as the IRR of a portfolio is not the same as the weighted average IRRs of individual investments.

Thus, using methodologies based on discounting at a constant rate is inadequate for the purposes of long-term investors

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who need performance measures that can help them make hedging, risk management, and portfolio management decisions.

The third point (the absence of unique pricing measures) is a reflection of what is usually labelled 'incomplete markets', i.e. the fact that the same asset can be valued differently by two investors, and yet this does not constitute an arbitrage opportunity (and therefore the bid-ask spread does not narrow) because transaction costs are high and, crucially, because different investors may value infrastructure assets for different reasons. For example, some may put a higher price on duration, while others may value inflation hedging. The diversification benefits of unlisted infrastructure and therefore its "fair" price also depend on investors' overall asset allocation and the size of their infrastructure bucket.

The existence of a range of values is also impacted by market dynamics: if a new type of investor (e.g. less risk averse) enters the long-term infrastructure equity market, the range of observable valuations for similar assets may change. Likewise, if some investors want to increase their allocations to unlisted assets, given the limited available stock of investable infrastructure projects at a given point in time, *their* valuations may rise, but not those of others (who may sell). Finally, if unlisted infrastructure equity returns can gradually be better hedged using traded assets, then individual subjective valuations should converge towards a unique pricing measure.

From this perspective, the oft-mentioned illiquidity premium expected by investors in unlisted assets is not a unique price. While *relatively* illiquid but traded instruments can yield a unique illiquidity premium, unlisted assets may command a different illiquidity premium for different types of investors.

This therefore leads to the important point that the required rate of return or discount rate of individual investors is **fundamentally unobservable**: it cannot be inferred from observable transaction prices since it is both a function of the characteristics of the asset (e.g. cash flow volatility) and individual investor preferences. Each observed transaction corresponds to a single pricing equation with two unknowns (project and investor characteristics) and cannot be solved directly.

Existing approaches

Existing approaches developed to value private equity investments are inadequate for the purpose of valuing unlisted infrastructure project equity.

In our review of the literature we identify three groups of valuation techniques: repeat sales, public market equivalents and cash flow driven approaches. These techniques all imply that enough data can be observed to compute a price. The repeat sales approach assumes that asset betas can be inferred from discrete and unevenly timed transaction observations after correcting for price staleness and sampling bias, while

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the public market equivalent approach implies that public asset betas can be combined to proxy the return of unlisted assets. Cash flow-driven approaches are less normative and simply aim to derive the unobservable rate of return of unlisted assets by decomposing their implied returns into traded and untraded components *ex post facto*, that is, once all cash flows have been observed and can be related to market factors.

Thus, these approaches cannot be directly applied to privately-held infrastructure investments, the value of which is determined by streams of expected and risky cash flows that mostly occur in the future, and for which few comparable realised investments exist today.

Existing approaches also typically fail to take into account the subjective dimension of asset pricing in the unlisted space and compute asset betas and alphas as if a unique pricing measure existed, i.e. as if all investors had similar preferences, and in some papers, as if private equity exposures could always be replicated with a combination of traded assets.

Proposed approach

To the extent that infrastructure dividend cash flows can only be partially observed, they cannot be decomposed into exogenous factors (markets, the economy, etc.), the future value of which is not known today and would be very perilous to predict 30 years from now.

Instead, we must derive the relevant discount factors *endogenously* i.e. by using observable information about each private investment in infrastructure equity including, as suggested above, its contractual characteristics, location, financial structure, etc., as well as the value of the initial equity investment made, which is also observable.

We therefore argue that a robust valuation framework for equity investments that solely create rights to future (and yet largely unobserved) risky cash flows, as is the case of privately-held infrastructure equity, requires two components:

1. A model of expected dividends and conditional dividend volatility, calibrated to the best of our current knowledge;
2. A model of *endogenously* determined discount factors, that is, the combination of expected returns *implied by the distribution of future dividends*, given observable investment values.

In other words, as for any other stock, the valuation of privately-held equity in infrastructure projects amounts to deriving the appropriate discount rates for a given estimate of future dividends. But while this process is *implicit* in the pricing mechanism of public stock markets, in the case of privately-held equity with distant payoffs, we have to derive the relevant parameters *explicitly*, taking into account the characteristics of infrastructure assets.

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Dividend distribution model & required data

The objective of the dividend model is to express and measure the distribution of future dividends, with a focus on data observable and available today, and with a view to determine a parsimonious data collection template allowing improved model calibration in the future.

For this purpose, and because of the empirical limitations highlighted above, we adopt a so-called Bayesian approach: we first build a *prior* distribution of the cash flow process at each point in the life of the investment, given the current state of knowledge about equity investments in infrastructure. Later, when new dividend data becomes available, this *prior* knowledge can be updated using Bayesian inference techniques to derive a more precise *posterior* probability distribution of dividends.

The option to update our knowledge at a later stage by setting up the cash flow model as a Bayesian inference problem, also allows us to determine what data needs to be collected today, which is one of the objectives of this paper.

We note that the fact that new observations are not redundant today (we can still learn a lot more about the dynamics of dividends in infrastructure investment by collecting data), **justifies the need for an ongoing and standardised reporting of these cash flows to keep learning about their true distribution** and value the infrastructure investments made today, tomorrow.

We argue that the dividend stream or cash flow process can be described as *state-dependent* and introduce a new metric for infrastructure project dividends – *the equity service cover ratio* or ESCR – which is computed as the ratio of realised-to-base case dividends.

The base case equity forecast of infrastructure equity investments, while not necessarily accurate, provides a useful and *observable quantity*, which by definition spans the entire life of each investment. Thus, we propose to describe the behaviour of equity cash flows in infrastructure projects *as a function of this initial forecast*, in order to create metrics allowing direct comparisons between different equity investments.

We show that the value of the ESCR at each point in the lifecycle of infrastructure equity investments can be used as a state variable describing the dynamics of the cash flow process. In combination with a given project's base case dividend forecast (which is known at the time of investment), knowledge of the distribution of the ESCR at each point in time is **sufficient to express the expected value and conditional volatility of dividends**.

The data required to implement this approach to modelling future dividends in infrastructure investments falls into three categories:

1. Individual investment characteristics needed to identify groups of infrastructure equity investments

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that can be expected *a priori* to have different underlying cash flow and dividend distributions between groups, and correspond to reasonably homogenous cash flow processes within groups;

2. Initial investment values and dividend base case forecasts at the time of investing and any subsequent revisions of these forecasts;
3. Actual investment values and realised dividend data needed to update ESCR distribution parameters at each point in the investment's life.

The required data is summarised in Table 3.

Valuation Framework

Since the term structure of expected returns of individual investors/deals is unobservable and lies within a range or bounds embodying market dynamics at a given point in time, we propose to adapt the classic state-space model mostly used in physical and natural sciences to capture the implied average valuation of the privately-held infrastructure equity market at one point in time and its change from period to period. Using such a model also allows us to capture the bounds on value implied by observable investment decisions for a given stream of expected cash flows.

Indeed, the objective of state-space models is parameter estimation and inference about *unobservable* variables in dynamic systems, that is, to capture the dynamics of observable data in terms of an unobserved vector – here the term structure of discount factors – known as the state vector of the

system (the market). Hence, we must have an *observation or measurement equation* relating observable data to a state vector of discount factors, and a *state or transition equation*, which describes the dynamics of this state, from one observation (transaction) to the next. The combination of the state and observation equations is known as a state-space representation of the system's dynamics.

Our unit of observation is the individual transaction, i.e. individual infrastructure equity investments. To each transaction corresponds a given stochastic dividend process characterised by a distribution of future cash flows and an initial investment value, both of which we assume to be *observable*.

Each transaction is the expression of a valuation "state", i.e. a given term structure of discount factors matching the price paid in that transaction (the initial investment) with expected cash flows. As discussed above, this state is *unobservable* because it is partly determined by the investor's subjective preferences and, in the absence of complete markets, cannot be discovered by pricing a portfolio of traded assets that would always replicate the investment's payoff.

Each transaction corresponds to a new state, i.e. a new valuation, which may or may not be the same than the previous transaction's. Given a stream of risky future dividends, if the price paid in the current transaction is different from that paid in the previous one, it must be because the valuation state

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has shifted. The valuation state can change due to a change in investor preferences between the two deals, or due to a change in the consensus risk profile of that kind of investments (e.g. projects with commercial revenues after a recession), or because of a change in the overall market sentiment (the average) valuation.

Thus, by iterating through transactions, we may derive an implied average valuation state (term structure of discount factors) and its range, bounded by the highest and lowest bidders in the relevant period.

Later, when dividend payments are realised, (conditional) per-period returns can be computed using the discounted sum of remaining cash flows as the end-of-period value (given the implied term structure of discount factors previously derived).

We define the *observation equation* using a dynamic version of the standard Gordon growth model (discounted dividends) and the *state equation* using an autoregressive model of the term structure of expected returns which can be derived from the kind of single or multi-factor models of expected excess returns that are commonly found in the literature.

In a simple, linear setting, we show that we can iterate through observable investments, while estimating model parameters on a rolling basis, to capture both the implied expected returns (and discount factors) during a given reporting period and track these values and their range (arbitrage bounds) from period to period.

Results

As an illustration of our approach, we apply the dividend and pricing models to a generic case of privately-held infrastructure investment, assuming an expected ESCR and ESCR volatility profile (including the probability of receiving no dividends ($ESCR = 0$) in any given period).

Given a base case dividend scenario inspired by an actual infrastructure project financed in Europe in the last decade, we obtain a full distribution of future dividends and apply our valuation framework to this assumed dividend process for a (or an equally assumed) range of investment values. Some of the key outputs are shown on the following figures.

Figure 1 shows the resulting filtered term structure of expected period and multi-period (average) expected returns filtered from a range of 20 initial observations (which could have happened during, say, one year).

Figure 2 shows the resulting values of the dividend discount factor (using continuously compounded (log) returns, the discount factor m is simply the exponent of minus the total return from the valuation date until the relevant period) at the time of valuation and the expected average price and its range for this group of transactions.

Finally, figure 3 shows how we can implement this model with rolling parameter estimation to track the implied average expected returns and price of consecutive transactions from period

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to period. In this example, the average price investors are willing to pay for the same infrastructure asset is assumed to increase continuously (perhaps because investors increasingly value assets that pay predictable dividends in bad states of the world), but the range of prices they are willing to pay to buy a stake in this (unchanged) dividend process is also assumed to change. Initially it is assumed to widen (say that new investors become active in this market and have different preference or views on risk); half way through the 200 observed transactions, the range of valuations is assumed to start shrinking (perhaps there is now a greater consensus amongst investors about risk or more traded assets allowing replication).

These results spring from model inputs that are only inspired by existing data and a number of intuitions about privately-held infrastructure equity investments, and can only be considered an illustration. However, they show clearly that with well calibrated cash flow models and a transparent valuation framework, the kind of performance measures that have so far been unavailable to long-term investors can readily be derived and monitored in time, as new investments are made.

Moreover, while the valuation of private assets using this framework, including the use of a term structure of expected returns, is only an application of a number of existing key principles of modern finance, the possibility to measure and track the evolution of the pricing bounds of such assets is an innovation in applied finance

which will allow investors and regulators to assess issues of pricing bubbles or measure the impact of regulatory change in a manner that was not possible so far.

Next steps

In this paper, we implement the first three steps on the roadmap for the creation of long-term infrastructure equity investment benchmarks defined in Blanc-Brude (2014a): focusing on well defined financial assets (as opposed to ill-defined industrial sectors), devising adequate pricing models based on modern asset pricing yet implementable given available data today; and determining a parsimonious set of data that can be collected and improve our knowledge of expected returns in privately-held infrastructure equity investments.

Next steps include the implementation of our data collection template to create a reporting standard for long-term investors and the ongoing collection of the said data. Beyond, in future research, we propose to develop models of return correlations for unlisted infrastructure assets in order to work towards building portfolios of privately-held infrastructure equity investments.

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Figure 1: Filtered estimates of the term structure of single period returns, $r_{t+\tau}$, are shown in the left panel, and multi period discount rates, $\mu_{t,\tau}$, after 20 transactions are shown in the right panel.

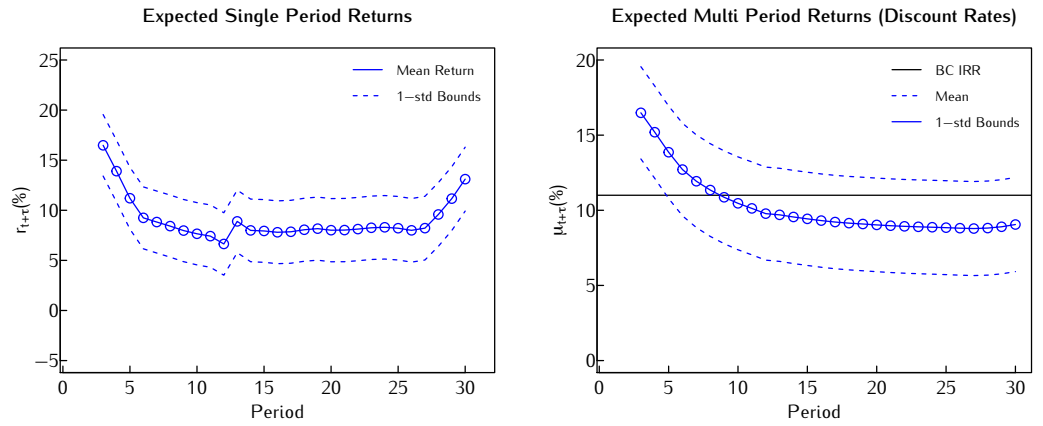


Figure 2: Stochastic discount factor, $m_{t+\tau}$, is shown in the left panel, and the expected evolution of equity price, P_{t_0} , is shown in the right panel.

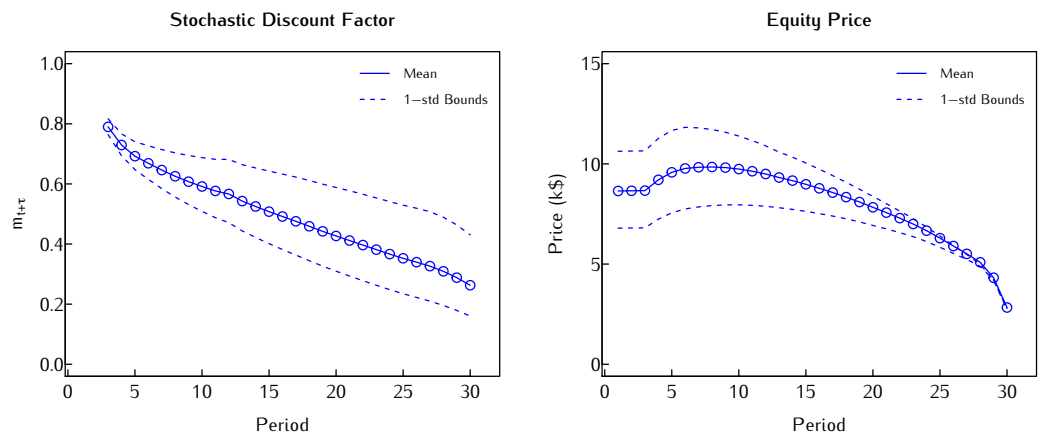
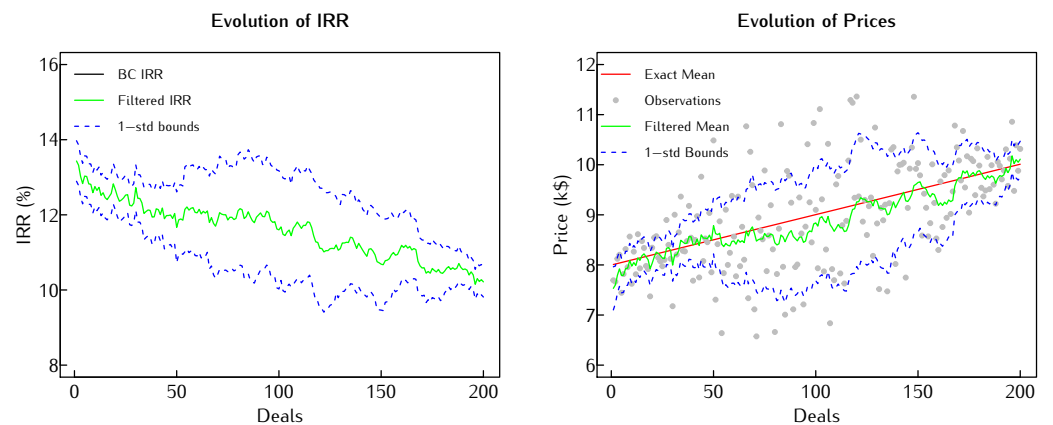


Figure 3: Filtered estimates of the IRR (left panel) and transaction prices (right panel), with one standard deviation bounds.



1. Introduction



1. Introduction

Matching the huge demand for capital in infrastructure projects around the world with the available supply of long-term funds by institutional investors, be they pension funds, insurers or sovereign wealth funds, has never been so high on the international policy agenda.

This momentum, illustrated by the recent focus on long-term investment in infrastructure by the G20, coincides with the steadily growing investment appetite from the same investors for unlisted and illiquid assets. However, fully fledged investment solutions demonstrating the benefits of privately-held equity investments in infrastructure have remained elusive and documenting their characteristics of has become a pressing question.

As a consequence, benchmarking the expected behaviour of long-term infrastructure investments would considerably help investors to fully integrate private infrastructure investment into their asset-liability management exercises.

In this paper, we aim to contribute to this debate by developing a **rigorous valuation framework for privately-held infrastructure equity investments**.

Our proposed approach is rooted in modern asset pricing and statistical inference theory, but remains a fully operational solution to the formation of performance expectations for sophisticated investors.

We also propose a parsimonious data collection template, which can be

used industry-wide to improve existing knowledge of the performance of privately-held infrastructure equity investments on an ongoing basis.

1.1 Objectives

We aim to achieve the following objectives:

1. Determine the most appropriate **valuation framework** for privately-held equity investments in infrastructure projects or entities;
2. Design a **methodology** that can be readily applied given the current state of empirical knowledge and, going forward, at a minimum cost in terms of data collection;
3. Derive some of the most relevant valuation and performance **measures** for long-term equity investors and regulators;
4. Define a parsimonious data collection **template** that nevertheless allows meeting the three points above.

1.2 The contractual nature of private infrastructure equity

We define infrastructure equity investments as privately-held shareholdings in firms created to build, operate and maintain certain infrastructure projects or networks. While such firms are often created specifically for the purpose of developing and operating a given infrastructure project for a finite time period, private infrastructure equity may also correspond to open-ended stakes in firms that own and

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operate multiple infrastructure assets, such as utilities.

As we have argued in a previous paper (Blanc-Brude, 2013), all such equity investments imply the creation of **relationship-specific** tangible assets. Whether they are directly owned by the firm delivering the infrastructure or not, relationship-specific assets are immobile and have no alternative uses. As a consequence, the value of the relevant equity investment can be said to be wholly determined by a series of long-term contractual arrangements, by which investors and their public or private clients agree to make large sunk investments and deliver a given service or output, in exchange for which they receive an income stream which is more/less risky for a sufficiently long period of time to recoup their initial investment. Equity investors in infrastructure are then the residual claimants to this income stream.

Thus, private infrastructure equity owners can be described as simply owning **rights to streams of future dividends** created by such contractual arrangements. Without such contracts and the commitment to a long-term relationship that they create, large sunk capital investments in relationship-specific infrastructure assets typically cannot take place.

In the immense majority of cases, these future streams of dividends also have a **finite life**, equal to the life of the relevant long-term contract. Hence, such investments typically have a *duration*,

which is an important component of the liability-friendliness of privately-held infrastructure equity investments, and should be of significant appeal to long-term investors, e.g. insurers or defined benefit pension plans.

Likewise, the volatility of infrastructure dividends is, in large part, determined by the network of contracts entered into by infrastructure firms. In particular, contracts can be used to control and transfer the cost of building, operating and maintaining an infrastructure project, as well as to determine each project's business model, e.g. the extent to which it receives a pre-agreed income from a unique client, or is exposed to commercial risks because it derives an income from multiple tariff- or toll-paying end-users.

Because of the role of long-term contracts in infrastructure investment, the various industrial sectors that are often associated with infrastructure (transport, energy, social, etc.) cannot be expected to explain much of the differences in valuations or expected returns. Instead, we expect systematic differences in the cost of equity (risk pricing) to be strongly correlated with the contractual features of private infrastructure assets.¹

Next, we discuss the empirical challenges that characterise the valuation of privately-held equity investments in infrastructure.

1 - Of course, factors external to the contractual setup of the firm can also be expected to have an impact on the variability of expected cash flows, such as different geographies (primarily contractual counterparty risk), the risk of technological obsolescence, etc.

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1.3 Empirical challenges

1.3.1 Data paucity

Two types of data can be used to value financial assets: transaction prices or cash flows. Transaction prices have an intuitive appeal since they are expected to embody the cumulative value of a stream of dividends discounted at the required cost of equity i.e. the implicit, market-driven result of one or other form of the seminal Gordon pricing model of dividend forecast and discounting (Gordon and Shapiro, 1956).

However, in the case of privately-held infrastructure equity, transaction price data is unlikely to be available in sufficiently large volumes: we can observe the initial investments of equity holders when new private infrastructure ventures are created, which may be interpreted as a price signal corresponding to a given expected dividend cash flow. However, unlike other types of unlisted investment such as venture capital, infrastructure project companies seldom lead to multiple financing rounds and even less frequently to IPOs, thus limiting the number of observable transactions. Secondary market sales of private infrastructure equity do occur, but in a context where such assets are mostly held to maturity by long-term investors, they are relatively rare, and observing such transactions is unlikely to yield representative samples of asset prices.

Indeed, if we were to estimate the determinants of private infrastructure equity transaction prices empirically, we would like to control for different types of risk factors explaining the *average*

difference in price between projects (i.e. the cross-section of prices), as well as the change of risk profile that we expect to see in numerous standalone projects characterised by the sequential resolution of uncertainty across their lifecycle, which requires observing times series of transaction prices for comparable investments.

Thus, observing *representative* samples of secondary market equity prices would require data at each point in the 20 or 30-year lifecycle of each type of infrastructure asset *in each (annual) reporting period*. Instead, observable samples of secondary market infrastructure acquisitions are likely to be affected by severe biases. Such biases are also compounded by the tendency of governments to roll out private investment opportunities in public infrastructure in bulk, in different periods and geographies, reducing the possibility of robust control groups when evaluating prices.²

Cash flow data on the other hand is more readily available. Most infrastructure projects, when they are financed, are the object of a dividend cash flow forecast or **base case scenario**, which spans the entire life of the investment and serves as the basis for the initial investment decision. The base case dividend forecast may vary between investors for comparable projects and substantially deviate from the true statistical expectation of dividends. The base case forecast can also be revised in subsequent periods as a function of realised states of the world, e.g. if realised dividends are substantially lower or higher

2 - Public infrastructure procurement goes through phases, e.g. real toll roads in Eastern Europe in the mid-1990s, shadow toll roads in Portugal in the early 2000s, "social" infrastructure PFIs in the UK between 1998 and 2006, etc. Private infrastructure procurement also goes through phases related to the business cycle, e.g. coal terminals in Australia during the past decade.

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than initially envisaged. Nevertheless, both initial and revised dividend forecasts made by infrastructure equity investors are *observable*, as well as, of course, actual dividends, as and when they occur.

1.3.2 The need to collect more data

Today a limited amount of base case and realised infrastructure dividend cash flow data has been aggregated. They are scattered amongst numerous private investors, and little or no effort has been made to construct a database of these cash flows. Building this database is a necessary step towards properly documenting the expected value and volatility of dividend cash flows in private infrastructure investment, and this paper aims to contribute to this objective by defining what data is required to implement a robust pricing model at the minimum data collection cost.

Nevertheless, it must be noted that today, *even with such a database*, empirical observations about infrastructure equity cash flows will remain truncated in time and limited in the cross-section.

First, observed dividend time series are incomplete: by definition, the immense majority of infrastructure currently investable by private equity owners is far from having reached the end of its life. Hence, most of these cash flows remain in the future for which very little, if any, comparable investments currently exist.

Indeed, in the cross section, the type of infrastructure projects that have been

financed in the past are not necessarily representative of investment opportunities today. For example, even if year-23 dividends for projects that were financed 24 years ago can be observed today, they may not be good predictors of dividends in projects financed 3 years ago, 20 years from now. For example, projects financed in the early 1990s may have been in sectors where fewer projects exist today, e.g. telecoms, or rely on contactual structures or technologies that are not relevant to long-term investors in infrastructure today, e.g. coal-fired merchant power.

Thus, any cash flow model that relies solely on past data, and ignores incoming, new information, may lead to significant estimation errors, that is, today **new infrastructure dividend data cannot be considered redundant** and its ongoing collection will be instrumental to the development of forward-looking performance measures that use all available information.

To sum up, while we can observe initial equity investment decisions in privately-held infrastructure, as well as base case and revised forecasts and – to some extent – realised dividend cash flows, **data paucity is an endemic dimension of the valuation of private long-term equity investments** and we must start from the premise that we cannot observe enough data to simply derive expected return measures empirically, from historical data.

Instead, we can build into our approach the possibility to improve or update our

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knowledge as new observations become available. And if these new observations are not redundant, **this justifies the need for an ongoing and standardised reporting of these cash flows to keep learning about their true distribution** and value the infrastructure investments made today, tomorrow.

1.4 Existing Valuation Approaches are Insufficient

Why do we need to develop a new valuation framework for private investors in infrastructure equity?

Existing research on private equity performance reporting overwhelmingly concludes that the self-reported net asset values (NAV), internal rates of return (IRR) and investment multiples reported by investment fund managers are both inaccurate and inadequate.

Inaccuracy springs from the tendency of PE managers to report their performance opportunistically (see Jenkinson et al., 2013, for a recent study). Phalippou and Gottschalg (2009) find a large negative correlation between duration and performance in private equity funds, which, combined with the incentive to time cash flows strategically, tends to create an upward bias in reported performance and creates incentives to exit investments quickly.

Meanwhile, PE performance metrics are *inadequate*: in their comprehensive critique of the performance monitoring

of typical private equity funds, Phalippou and Gottschalg (2009) find that pooling individual investments and funds IRRs also creates misleading results because IRRs cannot be averaged. Likewise, Jenkinson et al. (2013) find that current reported IRRs are poor predictors of the ultimate returns of PE funds.

Faced with unreliable reported valuation and performance measures, academic research on unlisted and illiquid equity investments has developed three types of approaches to measure performance: repeat-sales, public market equivalents and cash flow-driven approaches.

The first approach relies on the ability to observe discrete valuations in time. Initially developed for the real estate sector, the repeat-sales approach has been applied to venture capital and leveraged buy-outs using data from successive financing rounds, acquisitions or IPOs (see Woodward, 2004; Cochrane, 2005; Korteweg and Sorensen, 2007). However, as we suggested above, this methodology is unlikely to be applicable to infrastructure because unlike venture capital projects, infrastructure projects typically require only a single initial financing round and are then very infrequently traded.

Next, without sufficient feedback from transaction prices, the value of privately-held equity investments can be determined by discounting times series of expected cash flows.

1. Introduction

For example, studies such as Ljungqvist and Richardson (2003), Kaplan and Schoar (2005) or Phalippou and Gottschalg (2009) propose to calculate a public market equivalent (PME) by using the cash flows into and out of PE funds as if they represented buying and selling a public index, thus assuming a *beta* of *one*, and treat the resulting value as a measure of performance, with values higher than unity considered to indicate outperformance and *vice-versa*. However, these approaches do not allow the *beta* of private investments to be observed, but rather assume it can be proxied by an index of choice. If the implied *beta* is lower than the true *beta*, the measured outperformance is necessarily overstated and *vice-versa* (Woodward, 2004).

Another PME approach consists of matching private investments with industry betas, deriving the un-levered industry betas using industry averages and re-leveraging them using investment specific information (see Kaplan and Ruback, 1995; Ljungqvist and Richardson, 2003; Phalippou and Zollo, 2005, for various applications). Inspired by the CAPM, this approach relies on the use of observable market *betas* and average sector leverage information to derive un-levered equivalent *betas* and subsequently re-leverage them using project information. Esty (1999) and Cooper (2010) propose an application to infrastructure project finance.³

But such approaches assume that the project's equity *beta* is constant (only leverage changes), which is at odds with

the notion that the infrastructure equity risk profile may change with time with the sequential resolution of uncertainty – e.g. construction completion – hence, so would the covariance of infrastructure equity returns with market returns. Moreover, one of the important questions about private infrastructure equity investments is their ability to diversify public market risks. Picking a market *beta* from a universe of traded stocks answers this question *a priori*, controlling for leverage, which is trivial.

Next, recent developments in asset pricing and empirical techniques allow the cash flow streams of private equity investments to be decomposed into factors (risk premia and factor loadings) that are relevant to investors and their asset allocation questions. For example, two more recent papers aims to decompose PE fund cash flows as a function of public market movements i.e. instead of assuming a value of the asset *beta*, they propose *estimating* the *beta* of PE funds from observed fund cash flows (Driessen et al., 2012; Ang et al., 2013).

The implied returns from a series of cash inflows and outflows can be expressed in terms of the exposure to market factors they create, e.g. value, momentum, low volatility, etc., as well as the potential for a purely private premium, which can be positive or negative. But because of the documented bias in the reporting of NAV by PE funds discussed above, only samples of PE funds for which all cash flows can be observed i.e. that have reached maturity and have

3 - This approach involves estimating the *beta* of the private infrastructure projects' debt, which is not a trivial exercise (see Blanc-Brude et al., 2014, for a structural approach to measuring the performance of infrastructure project debt).

1. Introduction

returned all available funds to investors, can be used.

Thus, standard and advanced valuation techniques used for private equity funds, real estate or venture capital investments are ill-suited to meet the empirical challenges found in the valuation of privately-held equity investments in infrastructure. In particular, **these approaches all assume that enough (representative) price or cash flow data can be observed**, which may not be the case for privately held investments in infrastructure, especially time series of realised dividends spanning several decades, across a range of representative infrastructure *business models* that are also relevant to long-term investors today.

1.5 Valuing the Long-Term Today

To the extent that infrastructure dividend cash flows can only be partially observed, they cannot be decomposed into exogenous factors (markets, the economy, etc.), the future value of which is not known today and would be very perilous to predict 30 years from now.

Instead, we must derive the relevant discount factors *endogenously* i.e. by using observable information about each private investment in infrastructure equity including, as suggested above, its contractual characteristics, location, financial structure, etc., as well as the value of the initial equity investment made, which is also observable.

Hence, we argue that a robust valuation framework for equity investments that solely create rights to future (and yet largely unobserved) risky cash flows, as is the case of privately-held infrastructure equity, requires two components:

1. A model of expected dividends and conditional dividend volatility, calibrated to the best of our current knowledge;
2. A model of *endogenously* determined discount factors, that is, the combination of expected returns *implied by the distribution of future dividends*, given observable investment values.

In other words, as for any other stock, the valuation of privately-held equity in infrastructure projects amounts to deriving the appropriate discount rates for a given estimate of future dividends. But while this process is *implicit* in the pricing mechanism of public stock markets, in the case of privately-held equity with distant payoffs, we have to derive the relevant parameters *explicitly*, taking into account the characteristics of infrastructure assets.

1.6 Paper Structure

The rest of this paper is structured as follows: chapter 2 describes our approach to modelling dividend cash flows in private infrastructure, defines the data collection requirements to calibrate and update such models, as and when new data becomes available, and provides several examples.

Chapter 3 describes a valuation framework designed as a state-space model linking

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observable information about prices and expected cash flows to investors' unobservable discount factors, thus allowing the derivation of the implied term structure of discount factors in private infrastructure equity investments, for a given distribution of future dividends, at different points in time.

Chapter 4 illustrates our proposed methodology with a generic infrastructure equity investment.

Chapter 5 summarises and discusses our findings.

2. Expected Dividend Model



2. Expected Dividend Model

2.1 A Bayesian Framework

In this chapter, we develop a simple approach to modelling future dividends in privately-held infrastructure investments. As discussed in the introduction, our objective is to express and measure the distribution of dividends, with a focus on data observable and available today, and with a view to determine a parsimonious data collection template allowing improved model calibration in the future.

As we also discussed earlier, one of our premises is the absence of sufficiently large and representative datasets of realised dividends that span the entire cross-section and lifecycle of privately-held equity investments in infrastructure. Thus, frequency-based empirical techniques are effectively unavailable. Nevertheless, such endemic data paucity can be taken into account by approaching dividend forecasting through the lens of Bayesian statistical inference.

Bayesian inference starts from a position of relative ignorance and proposes to update current knowledge given what can be observed today and tomorrow. It allows the parameters of the distribution of interest (the dividend stream of an infrastructure investment in this case) to be treated as stochastic quantities, thus reflecting the limits of our current knowledge of these parameters. Hence, the variance of the distribution of, for example, the parameter representing mean expected dividends, represents our uncertainty about the true value of this parameter.

Thus, we must first build a *prior* distribution of the cash flow process at each point in the life of the investment, given the current state of knowledge about equity investments in infrastructure. Later, when new dividend data becomes available, this *prior* knowledge can be updated using Bayesian inference techniques to derive a more precise *posterior* probability distribution of dividends. We provide several examples of such techniques in this chapter.

Importantly, the option to update our knowledge at a later stage by setting up the cash flow model as a Bayesian inference problem, also allows us to determine what data needs to be collected today, which is one of the objectives of this paper.

Our intuition is that realised dividend payments in infrastructure projects are determined by different *states* of the underlying cash flow process. In particular, we know that investors' expectation of receiving strictly positive dividends is dependent on infrastructure projects having achieved a number of steps, from construction to operations, as well as being in a position where distributions to equity investors are both possible (e.g. the firm is not in default) and allowed (e.g. dividends are not "locked up" by lenders or regulators).

Thus, the dividend stream or cash flow process can be described as *state-dependent*. In the next section (2.2), we introduce a new metric for infrastructure project dividends – *the equity service cover ratio* or ESCR – which is computed as the ratio of realised-to-base case dividends.

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We show that the value of the ESCR at each point in the lifecycle of infrastructure equity investments can be used as a state variable describing the dynamics of the cash flow process. In combination with a given project's base case dividend forecast (which is known at the time of investment), knowledge of the distribution of the ESCR at each point in time is **sufficient to express the expected value and conditional volatility of dividends**. We introduce these cash flow metrics in more detail in the next section (2.2).

In the following sections (2.3 and 2.4), we propose a simple Bayesian setup to model and calibrate the distribution of the ESCR in time and thus compute expected cash flows for any given project corresponding to the same underlying ESCR process, given a base case dividend forecast. We also provide a illustrative examples of the *learning* process as new data is observed and the parameters of the true ESCR distribution are discovered.

Section 2.5 concludes and summarises the data required to implement and update a model of future dividends in privately-held infrastructure investments.

2.2 Cash Flow Metrics and Risk Measures

Our first insight into the dynamics of the private infrastructure dividend process comes from the decision to invest, which requires the determination of a dividend forecast or **base case**.

While the base case equity forecast only stands for *one possible scenario*, it is the *reference* scenario under which an investor decides to commit a certain equity amount at t_0 . Still, the base case is not necessarily the most likely scenario, and base case dividends do not necessarily correspond to the *expected value* of dividends. The accuracy of individual project dividend forecasts may also vary between investors and projects. Finally, as investors' observe realised cash flows in initial years, they may also revise their base case forecast.

However, given the data paucity discussed earlier, the base case equity forecast of infrastructure projects provides a useful and *observable quantity*, which by definition spans the entire life of each investment. Thus, we propose to describe the behaviour of equity cash flows in infrastructure projects *as a function of this initial forecast*, in order to create metrics allowing direct comparisons between different equity investments.

Two types of cash flow metrics can be computed as a function of the project equity base case: measures of the distribution of expected cash flows, and measures of the expected growth of dividends, which we discuss next.

2.2.1 Expected cash flows

In what follows, time t indicates a given point in time during the lifecycle of an investment project, typically the valuation standpoint, while time $\tau = 1, \dots, T$ indicates the remaining periods in the investment. This notation is useful to

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express expectations at a given point in time t , of future cash flows happening at time $t + \tau$. We will use the notations introduced in this chapter throughout the rest of this paper.

Next, for a stream of cash flow to equity $C_{t+\tau}^i$ received at $t + \tau$ with $\tau = 1, \dots, T$, in state of the world i , we can write:

$$ESCR_{t+\tau}^i = \frac{C_{t+\tau}^i}{C_{t+\tau}^0} \quad (2.1)$$

which is the **equity service cover ratio** (ESCR) at time $t + \tau$ for state i , with $i = 0$ referencing the base case scenario known at time t , and $C_{t+\tau}^0$ the series of base case dividends at time $t + \tau$. Hence, if realised dividend payments equal the base case, $ESCR_{t+\tau} = 1$.

As shown in appendix 6.1.1, the expected value of $ESCR_{t+\tau}$ is the expected tendency to diverge from the original dividend forecast; and the product of its expected value $E_t(ESCR_{t+\tau})$ with a given investment's base case cash flows at time $t + \tau$ in a given project is **the expected value of dividends** at time $t + \tau$ in that project, given the information available at time t .⁴

Likewise, we show in the appendix that the standard deviation of $ESCR_{t+\tau}$, or $\sigma_{ESCR_{t+\tau}}$, is a direct measure of the **conditional volatility** of dividends at time $t + \tau | t$.

2.2.2 Dividend Growth

As we shall observe in Chapter 3, equity pricing models typically use dividend growth rates to express cash flow forecasts. Expressing future dividends in terms of

dividend growth is also useful for privately-held infrastructure investments that, contrary to project finance, may not have a finite life such as utilities or airport and ports.

For this purpose, we introduce the *ESCR return* i.e. a measure of the change in ESCR between two periods or $ESCR_{t+\tau}/ESCR_t$, and the log of the ESCR return, written $escr_{t+\tau}$ (see appendix 6.1.2 for details).

We also define $G_{t+\tau}^i$ as the dividend growth rate in state i between time t and time $t + \tau$, and $g_{t+\tau}$ as the log of the growth relative $1 + G_{t+\tau}^i$.

We show in appendix 6.1.2 that $G_{t+\tau}$ can be written as a function of the base case dividend growth rate at time $t + \tau$ and the ESCR return between time t and $t + \tau$. Likewise, $g_{t+\tau}$ can be approximated by $g_{t+\tau}^0 + escr_{t+\tau}$ with $g_{t+\tau}^0$ the log of the *base case dividend growth rates*. The same relations approximately hold in expectation as shown in the appendix.

Thus, given an investment base case, **knowledge of the distribution of $ESCR_{t+\tau}$ is sufficient to express the expected value and volatility of dividends, as well as their expected growth rate.**

In chapter 3, we show that equity investors' implied discount rates in infrastructure projects can also be derived from the combination of a pricing equation expressed as a function of $E_t(ESCR_{t+\tau})$ and a discount rate term structure equation expressed as a function of $\sigma_t ESCR_{t+\tau}$.

4 - We note that the ESCR cannot be computed if dividends are paid when zero dividend was forecasted in the base case in the same period. While this situation would not impact the valuation of the investment it would not allow the calibration of ESCRT to be improved for that particular dividend period in the lifecycle. However, with annual reporting periods and using investors' latest base case forecast, we expect this case to be rare.

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Finally, we highlight the role of the $ESCR_t$ to help measure the impact of *construction risks*.

2.2.3 Controlling for Greenfield Risks

Infrastructure investment only pay dividends after they have been built and become operational i.e. once they complete their so-called *greenfield* phase. Greenfield risks add conditionality to the stochastic equity distributions of infrastructure projects in two ways: first, the project may cost more to build than originally anticipated i.e. capital costs may increase (but not expected dividends) and, second, project development may suffer delays and thus the first dividend may not occur in the period originally envisaged.

To control for **construction cost overruns**, we introduce the normalised equity service cover ratio or $nESCR$, which is concerned with those construction cost overruns that have to be borne by equity investors. In this case, additional equity capital is injected into the project company.

The normalised $ESCR_t$ accounts for this: dividing the base case cash flow by the base case investment, and the cash flow in state i by the initial investment in the same state, so that:

$$nESCR_t^i = ESCR_t^i / \frac{C_0^i}{C_0^0}$$

With C_0^0 the initial equity investments in the base case and C_0^i the initial equity investments in state i (see appendix for details).

If there is no construction risk, $nESCR_t = ESCR_t$. For simplicity we only refer to $ESCR_t$ in what follows. When implementing the model, we use $nESCR_t$ to control for any increase in initial capital costs.

Project completion delays create a different type of uncertainty. For example, say that at t_0 in the base case of a greenfield project, equity investors expect receiving a first dividend at t_5 . As we show in chapter 3, investors hold a term structure of discount rates which applies to each future cash flow in the base case, here from t_5 onwards. However, the project may be delayed and the first cash flow only occur in, say, t_6 . Thus, the *potential impact of delays is measured by the probability of receiving a dividend given that no dividend was received until then*, in other words, the probability of the project moving from its construction or development state to its operational and dividend paying state, conditional on not having made that transition in previous periods.

We develop this point in the next section, in which we describe how we can model and calibrate the transition of $ESCR_t$ from a non-payment to a payment state, using a simple procedure relying on easily observable data.

2.3 Dividend State Transitions

At each point in time during the life of a private infrastructure equity investment, we can think of the dividend process as being in either one of two states: a zero-dividend state or strictly positive dividend state.

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Zero dividend payments can occur for different reasons at different points in time. For example, during the initial construction period, dividend payments are usually not possible. Later on, dividend payments can be suspended if the project company is in a state of default, or because of a dividend "lockup" following a covenant breach under the senior debt contract. In most cases, dividend payments eventually resume, even though the dividend forecast may then have to be revised.

Importantly, **these states are all observable** and are or can be recorded as part of the ongoing monitoring of long-term investments conducted by investors and their managers. Thus, we know that such observations can be used to calibrate a **model of state transitions**

The unconditional distribution of $ESCR_{t+\tau}$ is thus likely to be bimodal (i.e. to have two peaks): at each point in the investment's life there is some probability that no dividend will be paid, in which case $ESCR = 0$, otherwise some multiple of the base case dividend is paid, and $ESCR \in]0, +\infty[$.

Conditional on being in the positive dividend state, the expected value of dividends at time $t + \tau$ is equal to the product of the expected value of $ESCR_{t+\tau}$ in that state with the equity base case at that time.

It follows that we can model $ESCR_{t+\tau}$ as the combination of the probability of being in the strictly positive dividend state with the distribution of dividends in that state. We

discuss the former in the rest of this section, and the later in section 2.4

2.3.1 Payment Expectations

Say that dividend payments can take two states P_t at time t : a strictly positive payment, denoted by $P_t = 1$, or no payment denoted by $P_t = 0$.

The probability of observing a strictly positive dividend C_t is defined as $Pr(P_t = 1) = p_t$ while $Pr(P_t = 0) = q_t = 1 - p_t$. It follows that p_t is also the probability that $ESCR_t$ be strictly positive since,

$$\begin{aligned} p_t &= Pr(C_t > 0) \\ &= Pr\left(\left[\frac{C_t}{C_t^0} = ESCR_t\right] > 0\right) \end{aligned}$$

Furthermore, future payment states can be modelled as a function of the current state. Denoting time $i = \tau - 1$, for $j, k = 0, 1$, let $\pi_{jk} = Pr(P_{t+\tau} = k | P_{t+i} = j)$ be the state transition probabilities, with the one-step transition probability matrix given by:

$$\mathbb{P}_{t+i} = \begin{pmatrix} \pi_{11} & \pi_{10} \\ \pi_{01} & \pi_{00} \end{pmatrix}$$

Here, π_{11} is the probability of observing a strictly positive dividend at time $t + \tau$ conditional on having observed a strictly positive dividend at time $t + i$, and π_{10} is the probability of observing a zero dividend at time $t + \tau$ conditional on having observed a strictly positive dividend at time $t + i$.

The probability of observing a dividend at $t + \tau$ conditional on the realised state at $t + i$ is

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thus written:

$$\rho_{t+\tau} = \rho_{t+i}\pi_{11} + (1 - \rho_{t+i})(1 - \pi_{00}) \quad (2.2)$$

And in the matrix notation

$$\begin{bmatrix} \rho_{t+\tau} \\ q_{t+\tau} \end{bmatrix} = \mathbb{P}_{t+i} \cdot \begin{bmatrix} \rho_{t+i} \\ q_{t+i} \end{bmatrix} \quad (2.3)$$

That is, the probabilities of being in the payment (non-payment) state in period $t + \tau$ are determined by the product of the transition matrix with the probabilities of being in the payment (non-payment) state in the previous period $t + i$.

Hence, starting from any point in time, in which we know which state the ESCR is in (i.e. ρ_t is either 1 or 0), we can compute the probabilities of being in the payment and non-payment states at future periods by successively applying the transition matrix.

Thus, in expectation at time t , $ESCR_{t+\tau}$ can be written:

$$\begin{aligned} E_t(ESCR_{t+\tau}) &= \begin{pmatrix} \rho_{t+\tau} \\ q_{t+\tau} \end{pmatrix}^T \cdot \begin{pmatrix} E_t[ESCR_{t+\tau} | \rho_{t+\tau}=1] \\ 0 \end{pmatrix} \\ &= \rho_{t+\tau} E_t[ESCR_{t+\tau} | \rho_{t+\tau}=1] \end{aligned} \quad (2.4)$$

According to equation (2.3), we can know the conditional probabilities of receiving a strictly positive dividend in each future period $t + \tau$ by estimating \mathbb{P}_{t+i} across the project lifecycle for $i = 0, \dots, (T - 1)$, as well as initial payment state conditions.⁵

For example, in the context of a greenfield (new) infrastructure project, **initial**

conditions at t_0 are set to $\pi_{11} = \rho_0 = 0$ and $\pi_{00} = q_0 = 1$.

Note that estimating such transition probabilities can address the question of the timing of the first dividend and the possibility of construction delays, as discussed in section 2.2.3. By definition, no dividend is paid until the end of the construction period, which at t_0 is expected to last for, say, c periods.

The Markov chain thus starts with **the first expected dividend** so that $\tau = c, \dots, T$, with $c < T$, and

$$\mathbb{P}_{t+c|t} = \begin{pmatrix} 1 & 0 \\ \pi_{01} \geq 0 & \pi_{00} \leq 1 \end{pmatrix}$$

That is, π_{01} , the probability of moving to payment state 1 (positive dividend) at time $\tau = c$ given information at time t , is positive. Nevertheless, the timing of the first cash flow is uncertain, and π_{00} , the probability of staying in the zero dividend state at that time, is not null in $\mathbb{P}_{t+c}, \mathbb{P}_{t+c+1}$, etc.

Thus, we can incorporate the expected impact of construction delays on the timing of the cash flow process by estimating the value of π_{01} in \mathbb{P}_{t+c} .

Next, we show how we can estimate these state transition probabilities at each point in time using a simple Bayesian framework.

5 - This Markov chain can be extended to include an *absorbing state* (with a probability of staying in that state at the next period of 1) representing bankruptcy, as suggested above. We ignore this rare case here for simplicity.

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2.3.2 Estimating transition probabilities

By definition, the values of any $\mathbb{P}_{t+\tau}$ are such that $\pi_{11} + \pi_{10} = 1$, hence for example:

$$\mathbb{P}_t = \begin{pmatrix} \pi_{11} & 1 - \pi_{11} \\ \pi_{01} & 1 - \pi_{01} \end{pmatrix}$$

That is, each row of $\mathbb{P}_{t+\tau}$ matrices is equivalent to an independent Bernoulli draw of parameter π_{jk} , and we only need to estimate π_{11} and π_{01} to know the entire transition matrix at time $t + \tau$.

For example, with an observable population of N projects – the number n of successful draws corresponding to observing n strictly positive dividend at time $t + \tau$ given that we also observed a strictly positive dividend at the previous period – follows a Binomial distribution $Binomial(\pi_{11}, N)$, with the likelihood:

$$\begin{aligned} \mathcal{L}(\pi_{11}; n, N) &= \rho(n|\pi_{11}) \\ &= \binom{N}{n} \pi_{11}^n (1 - \pi_{11})^{N-n} \end{aligned}$$

As we detail in the appendix (6.4), we can give a Beta prior density to π_{11} , by which $Pr(\pi_{11}) = Beta(\alpha, \beta)$. This prior is said to be *conjugate* with respect to the Binomial likelihood of the data so that:

$$Pr(\pi_{11}|n, N) = Beta(\alpha + n, \beta + N - n) \tag{2.5}$$

In other word, by assuming that the true value of π_{11} is the mean of a Beta distribution of parameters (α, β) (known as the meta-parameters), given that the

likelihood function of the data follows a Binomial distribution of parameter π_{11} with N data points, we can update the values of the meta-parameters each time we observe n successes amongst N new data points.⁶

The posterior distribution of π_{11} summarises the state of our knowledge by combining information from newly available data expressed through the likelihood function, with *ex ante* information expressed through the prior distribution.

Each time new observations are made (N projects, n transitions from state 1 to state 1), our knowledge of the meta-parameters of the probability distribution of π_{11} can be updated and the accuracy of its distribution improved. The posterior distribution of $Pr(\pi_{11})$ then becomes a *new prior* each time new empirical observations become available. Bayesian inference thus allows sequential learning about the expected behaviour of infrastructure project equity cash flows.

The same process is used to estimate π_{01} .

2.3.3 Example of Bayesian learning of dividend state transition probabilities

In this section, we provide an illustration of this "learning" process following the observation of the required data, and how fast the true probabilities characterising dividend cash flows may be known.

We begin by assuming that the dividend process of a given infrastructure investment is characterised by the *true* transition

6 - In short: if $\pi \sim Beta(\alpha, \beta)$, $n \sim Binomial(\pi, N) \Rightarrow \hat{\pi}|n, N \sim Beta(\alpha + n, \beta + N - n)$

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Table 1: True, prior and posterior values of example transition probabilities at time t

	π_{11}	π_{01}
true value	0.9	0.95
prior value	0.8	0.8
posterior value	0.8941	0.9508
Prior variance	0.15	0.15
Posterior variance	0.00008	0.00004
Variance reduction	~3,800%	~1,900%

Table 2: True, prior and posterior values of example dividend distribution parameters at time t

ESCR Values			
	mean μ_{ESCR_t}	std dev σ_{ESCR_t}	precision*
true value	0.9	0.3	11.11
prior value	1	0.5	4
posterior	0.8955	0.3267	9.4
Lognormal param. $ESCR \sim \text{LogN}(m, \rho)$			
	location m	std dev σ	precision ρ
true value	-0.1580	0.3246	9.4912
prior value	-0.1115	0.4723	4.4814
posterior	-0.1637	0.3267	9.3701
Meta-param. $m \sim N(\mu, \delta)$			
	mean μ	precision δ	
prior value	m prior	1	
posterior	-0.1637	601	
Meta-param. $\rho \sim \Gamma(a, b)$			
	mean μ_ρ	std dev σ_ρ	
prior	ρ prior	50	
posterior	9.3701	0.3789	
	shape (a)	rate (b)	
prior value	0.8033	0.1792	
posterior	300.8033	0.0156	

* precision is the inverse of the variance

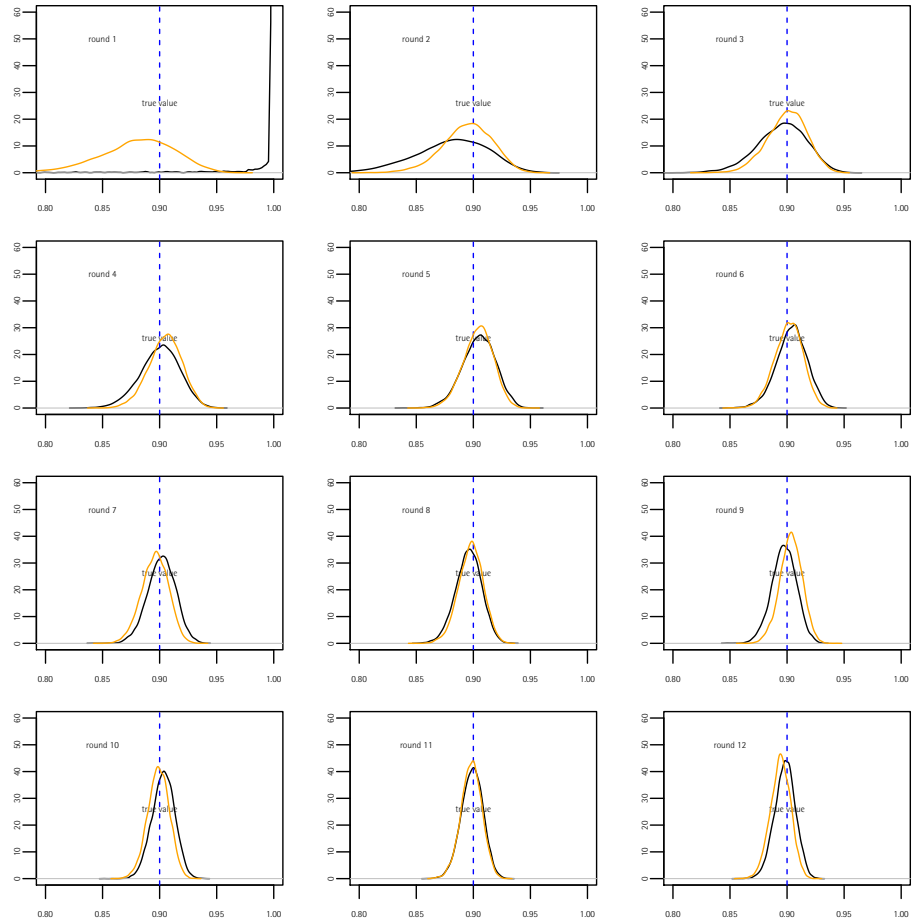
probabilities at some time t given in table 1. In this case, the true value of $\pi_{11} = 90\%$ and that of $\pi_{01} = 95\%$ i.e. in this type of infrastructure project, if a strictly positive dividend was paid at the previous period there is 90% chance of receiving a strictly positive dividend during the current period, and if no dividend was paid at the previous period then the chances of getting paid during the current period is even higher at 95%. This could correspond to a mature infrastructure project paying predictable cash flows and unlikely to experience zero dividends for long.

The same table also reports a prior value for π_{11} and π_{01} , which, before observing any data, we set at the conservative estimate of 80% for both values. Since the prior expected value of π_{11} (π_{01}) is 0.8, the beta distribution of π_{11} (π_{01}) also has mean of 0.8. The prior variance of both parameters is set at 0.15, which is very close to the maximum variance possible given the prior mean.⁷ Hence, we assume the values of π_{11} and π_{01} largely unknown around their otherwise conservative prior mean estimate.

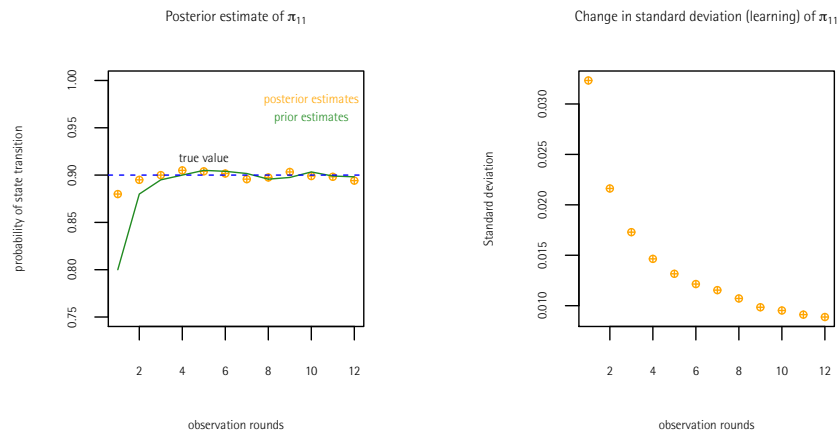
7 - It can easily be shown that the variance of a random variable following a Beta distribution of mean μ must be strictly inferior to $\mu(1 - \mu)$.

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Figure 4: Prior and posterior densities of the Beta distribution of π_{11} over 12 iterations using 50 data points

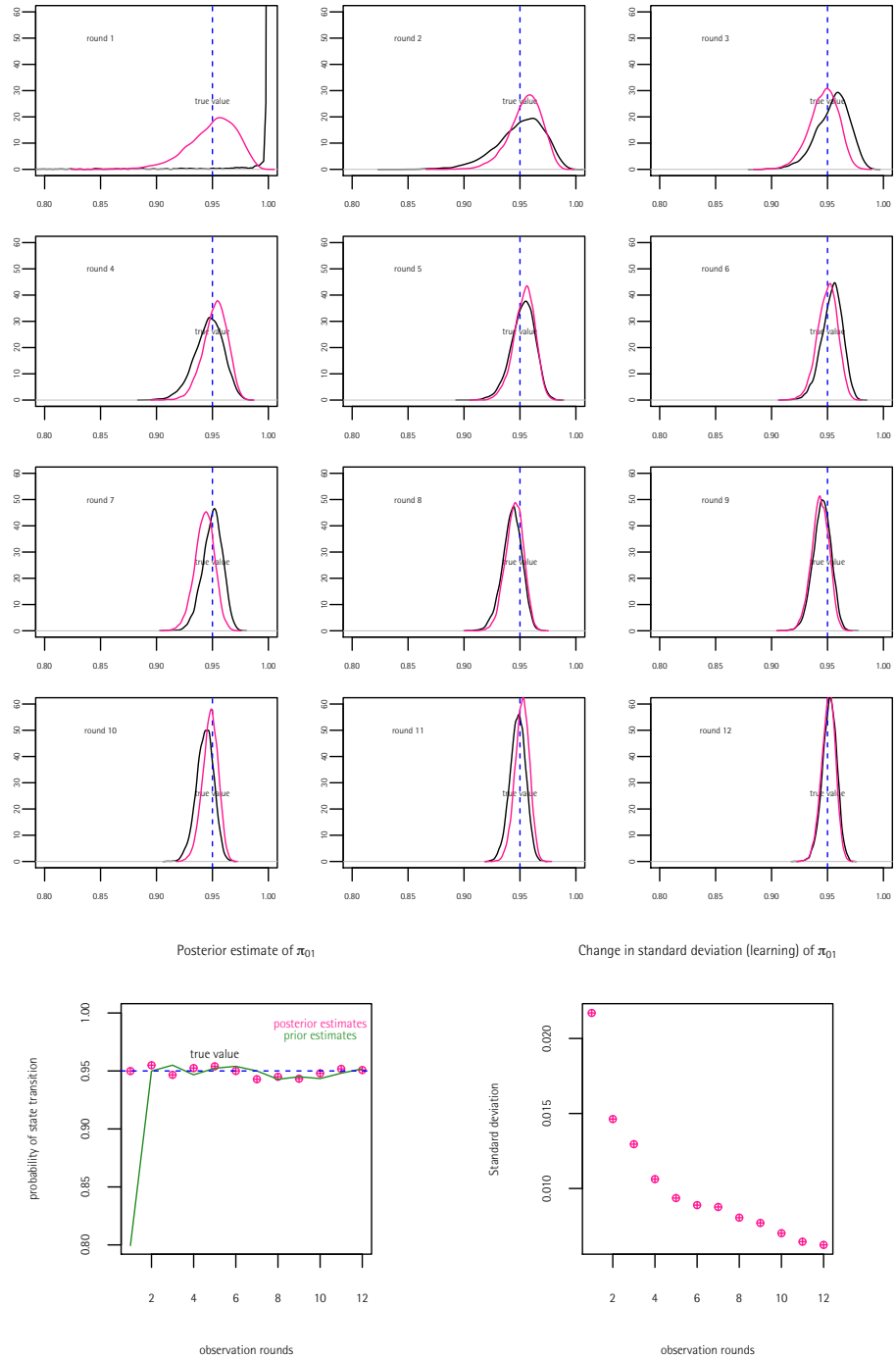


(Above) In each round of new observations, the prior density is indicated in black and the posterior in orange. Note that each posterior becomes the prior of the next round. The dotted blue line indicates the true value of the parameter being estimated.
 (Below left) In each observation round the green line indicates the prior and the pink dot the posterior value.



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Figure 5: Prior and posterior densities of the Beta distribution of π_{01} over 12 iterations using 50 data points



2. Expected Dividend Model

From these prior mean and variance of the two π parameters, which are simply *ad hoc* guesses on our part, we can derive the prior values of the meta-parameters α and β of their Beta distribution from the definition of the mean and variance of Beta-distributed variables, as shows in appendix 6.4.1.

Next, we generate 12 random samples of observations that follow the true binomial distribution of the observable data (the number of strictly positive dividends observed given the previous state) for a sample of $N = 50$ projects, using a computation software (R 3.2), and update the values of the meta-parameters according to the rule described in equation 2.5.

The new or *posterior* values of α and β can be used to compute the posterior mean and variance of each π parameter. In this example, we repeat this procedure 12 times to show the extent and pace of the learning process afforded by Bayesian inference.

Figures 4 and 5 illustrate this updating process by which each posterior value becomes a new prior, each time new data can be observed. The mean or expected value of each π rapidly converges towards its true value, and the variance of each parameter π rapidly decreases, indicating that their values are known with more and more certainty.

Finally, estimates of π_{11} and π_{01} can be used to compute the probability of receiving a positive dividend at time t following equation 2.2, given a value of p_{t-1} (which

has to have been computed first, or can be the initial condition when the project begins e.g. $p_0 = 0$ with certainty), as shown on figure 6.

It should be noted that most of the "learning" occurs during the first few rounds of observations, during which the expected value of parameters π_{11} and π_{01} rapidly converge towards their true value and their variance decreases equally rapidly. Because we started with a high level of variance for each value of π , the learning process from observing even the first few rounds of data is extensive and variance reduction very significant. These results suggest that observing the first five rounds of 50 projects is sufficient to achieve parameter estimates that are very close to their true values, assuming that the nature of the underlying process does not change.

It must also be stressed that learning could take place more rapidly if the prior was more "informed" i.e. the product of a robust *ex ante* model of project cash flows instead of the *ad hoc* intuitions that were used here.⁸

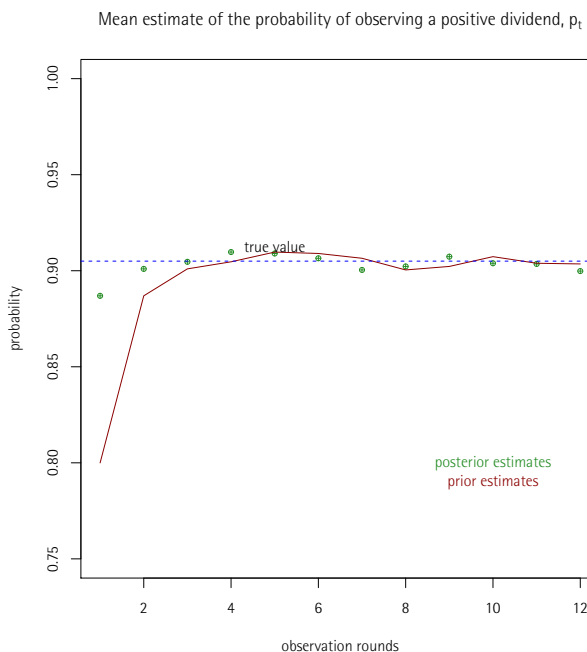
2.4 (conditional) Dividend Payment Distribution

Following the expression of the expected value of $ESCR_t$ given in equation 2.4, once the probability $p_{t+\tau}$ of being in the strictly positive dividend payment state at time $t + \tau$ is known or calibrated as best as current information allows, the distribution of dividends in the payment state still needs to be estimated.

8 - We propose such a model in the case of non-recourse infrastructure project finance using a powerful calibration tool relying on debt service cover ratios in Blanc-Brude et al. (2014).

2. Expected Dividend Model

Figure 6: Prior and posterior estimates of the probability of observing a positive dividend at time t



9 - In Blanc-Brude (2014b), we show empirically that the debt service cover ratio (DSCR) in non-recourse infrastructure project finance is log-normally distributed, and since the free cash flow of the firm is simply the product of the DSCR with the debt service (a constant), we can reasonably argue that the free cash flow of the firm and its dividends have a similar function form.

Conditional on being in the positive dividend state, $ESCR_{t+\tau} \in]0, +\infty[$. If we can give $ESCR_{t+\tau}$ a functional form, then it can be calibrated using an informed prior, as we suggest below, and later updated when new data can be observed. Conversely, if no functional form is assumed for the distribution of $ESCR_{t+\tau}|P_{t+\tau}=1$, numerical sampling techniques (e.g. Markov Chain Monte Carlo) can be used to update our knowledge of the distribution of dividends.

In what follows, we show how we can easily to derive the meta-parameters of a distribution of dividends that follow a lognormal process. Again, this assumption is not necessary but allows us to present a simple illustrative example.

Moreover, as we show in Blanc-Brude (2014b), there are good reasons to believe that cash flows are indeed log-

normally distributed at least in the case of infrastructure project finance.⁹

For instance, say we can observe N instances of strictly positive dividends at time t , and know the base case dividend, so that we can compute the data $X_n = ESCR_{n,t}$, for $n = 1, \dots, N$.

If X is considered to follow a lognormal process of mean (location) m and precision p , then its likelihood function is given by:

$$\mathcal{L}(m, p|X) \propto p^{N/2} \exp\left(-\frac{p}{2} \sum_{n=1}^N (\ln(X_n) - \mu)^2\right) \tag{2.6}$$

Fink (1997) and others show that the conjugate prior of a Lognormal process is a Gamma-Normal distribution, that is, as a function of m and p , equation 2.6 is

2. Expected Dividend Model

proportional to the product of a Gamma function of p (with parameters a and b) with a Normal distribution (with mean μ and precision δ) of m conditional on p .

The conjugate prior (for $p > 0$) is written:

$$Pr(m, p|a, b, \mu, \delta) = \frac{p^{a-1} \exp(-\frac{p}{b})}{\Gamma(a) b^a} \left(\frac{p\delta}{2\pi} \right)^{\frac{1}{2}} \exp\left(-\frac{p\delta}{2}(m - \mu)^2\right)$$

2.4.1 Parameter estimation

As we detail in the appendix (6.4.2), the sufficient statistics (required observations) to update the distribution are the number of observations N , the mean of the log data $\bar{X} = \frac{\sum_{n=1}^N \ln(X_n)}{N}$, and the sum of squared deviation of the log data about m (SS).

The joint posterior distribution $Pr(\hat{m}, \hat{p})$ is then given by the meta-parameters:

$$\begin{aligned} \hat{a} &= a + \frac{N}{2} \\ \hat{b} &= \left(\frac{1}{b} + \frac{SS}{2} + \frac{\delta N (\bar{X} - \mu)^2}{2(\delta + N)} \right)^{-1} \\ \hat{\mu} &= \frac{\delta \mu + N \bar{X}}{\delta + N} \\ \hat{\delta} &= \delta + N \end{aligned}$$

We discuss in the appendix how we may derive priors for the value of the meta-parameters a , b , μ and δ from the initial priors of the parameters of the log data, m and p .

Once, the values of N , \bar{X} and SS have been observed, updated (posterior) meta-parameters can be computed directly using the formulas above and a new parametrisation (\hat{m}, \hat{p}) of the distribution

of $ESCR_t$ is obtained that incorporates both prior knowledge and the new information.

As before the values of \hat{a} , \hat{b} , $\hat{\mu}$ and $\hat{\delta}$ become the new priors each time new observations of $ESCR_t$ are made

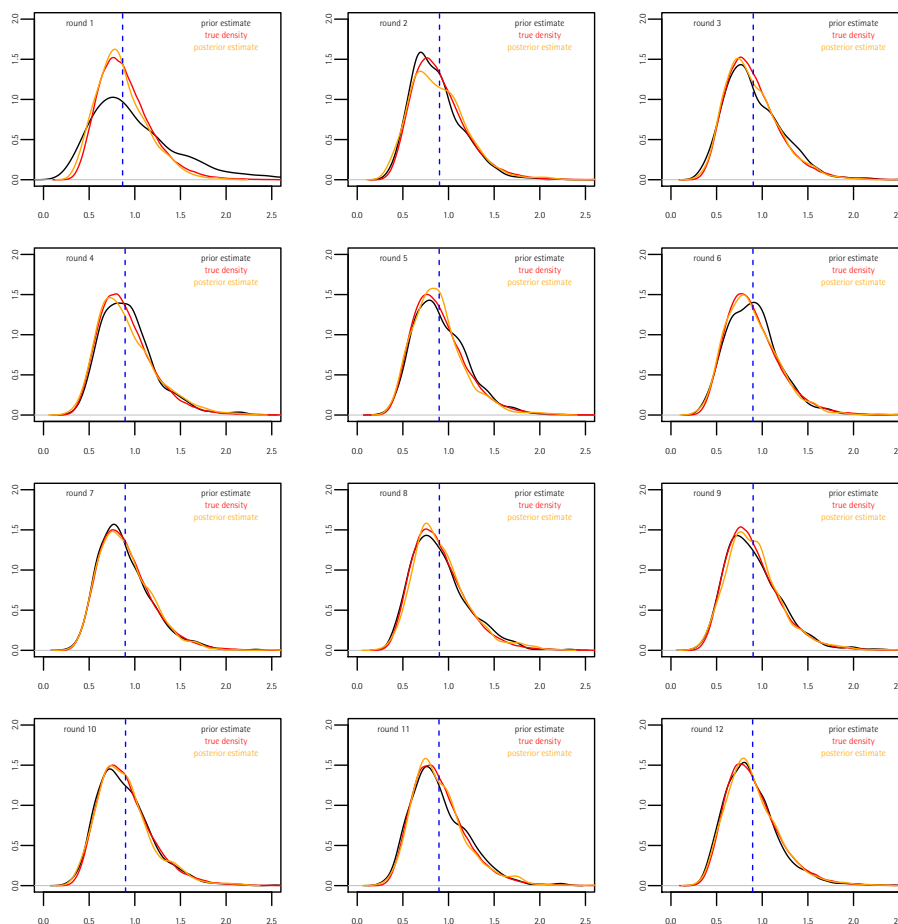
2.4.2 Example of dividend distribution calibration

In this section, we provide an illustration of the updating process as new ESCR observations are made. As before, we assume a true conditional distribution of $ESCR_t$ as described in table 2. Here, the true value of mean $ESCR_t$ in the payment state is 0.9 with a standard deviation of 0.3. The initial priors for these two parameters are set thus: the base case is (incorrectly) expected to be realised on average and the prior mean ESCR is set to 1, however this expected outcome is considered rather volatile and prior standard deviation of ESCR at time t is set at 0.5.

As detailed in the appendix, starting from these prior values of the arithmetic mean and variance of $ESCR_t$, a set of prior parameters of the log data (location m and precision p) are derived and a further set of prior values of the meta-parameters (mean μ and precision δ of the Normal distribution of m ; and shape a and scale b of the Gamma distribution of p) are also derived using the fundamental definitions of the relevant density functions and setting the unknown variance of m and p to large numbers, indicating significant parameter uncertainty.

2. Expected Dividend Model

Figure 7: Prior and posterior densities of the lognormal distribution of $ESCR_t$ over 12 iterations using 50 data points



The dotted blue line indicates the true arithmetic mean of the distribution and the red line its true density. In each round of new observations, the prior density is indicated in black and the posterior in orange. Note that each posterior becomes the prior of the next round.

The priors used in this example are also reported in table 2.

Next, we generate 12 rounds of $N = 50$ observations drawn from the true conditional distribution of $ESCR_t$, and apply the updating procedure described above.

Figures 7, 8 and 9 illustrate the extent of the "learning" as more data is observed and the prior distributions of m and p converge towards the true parameter values and their variance is reduced as the true values are gradually revealed.

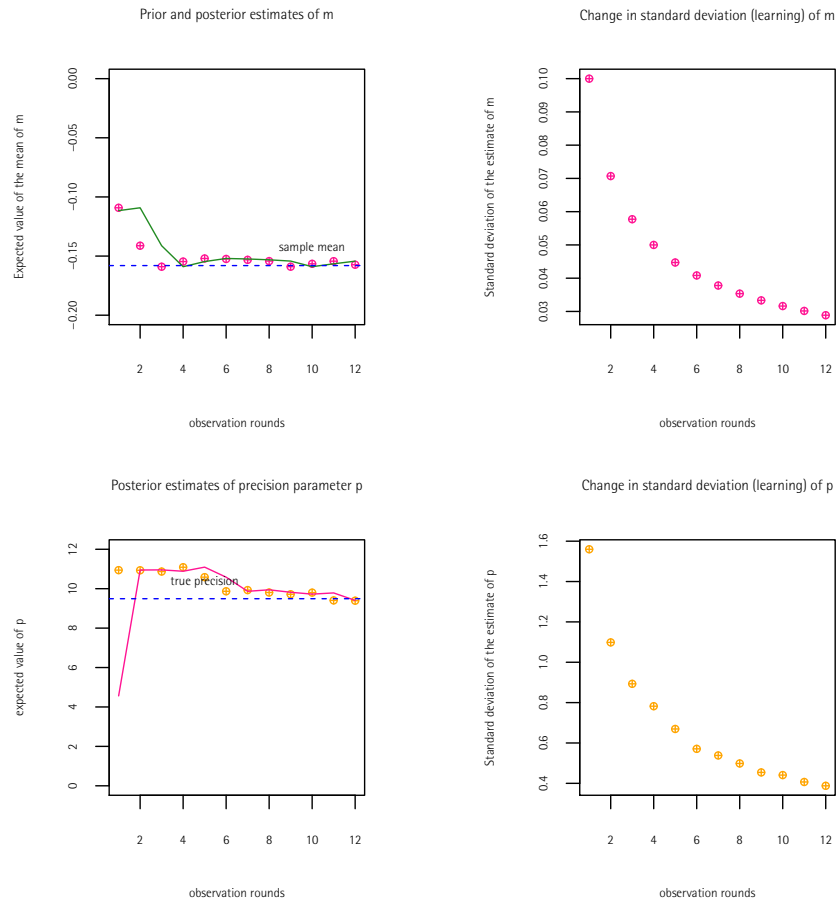
As before, the pace of learning is marginally decreasing i.e. most of the relevant information about the true parameters of the distribution of $ESCR_t$ is revealed during the initial rounds of observations. After the 5th iteration, parameter estimates have converged towards values very close to the true underlying values, with minimal variance (each iteration is reported in the appendix).

2.5 Data Collection Requirements

Thus, following a Bayesian approach, we can build a prior distribution of $ESCR_t$

2. Expected Dividend Model

Figure 8: Prior and posterior densities of the lognormal distribution of $ESCR_t$ over 12 iterations using 50 data points



Solid lines indicate the prior in this each round and dots the posterior, the dotted blue line is the true value of the parameter.

in privately-held infrastructure investments and update it as and when more data becomes available. Figure 10 provides a summary of the process described above.

Combined with the most recent base case cash flow forecast of a given equity investment, knowledge of the distribution of $ESCR_t$ provides us with a view on expected future dividends and the volatility of future dividends for this investment.

2.5.1 Prior elicitation

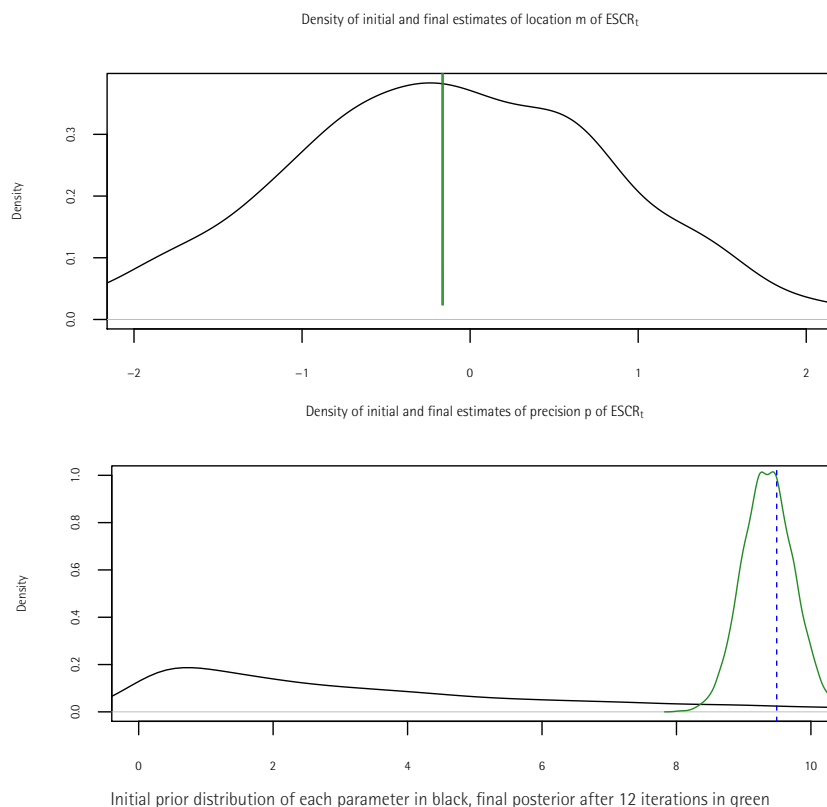
However, it goes without saying that infrastructure equity investments cannot

all be considered to draw their dividends from the same underlying distribution. Instead, this process of discovery of the distribution of future dividends may be applied selectively to different sub-groups of privately-held equity investments in infrastructure, which we expect to correspond to a homogenous underlying dividend process.

For instance, we expect the risk profile of limited-recourse infrastructure project financings to be determined by the contracts entered into by the project company, in particular the presence of

2. Expected Dividend Model

Figure 9: Prior and posterior densities of the m and p parameters of the lognormal distribution of $ESCR_t$



commercial risks (tolls), and the financial structuring decisions taken at the onset of the project (financial close) (see Blanc-Brude, 2013; Blanc-Brude et al., 2014, for a discussion).

Thus, we may form different priors for investments corresponding to different types of contracts or financial structures. Project financed infrastructure investment also have a finite life and thus require equally finite forecasts of dividend payouts.

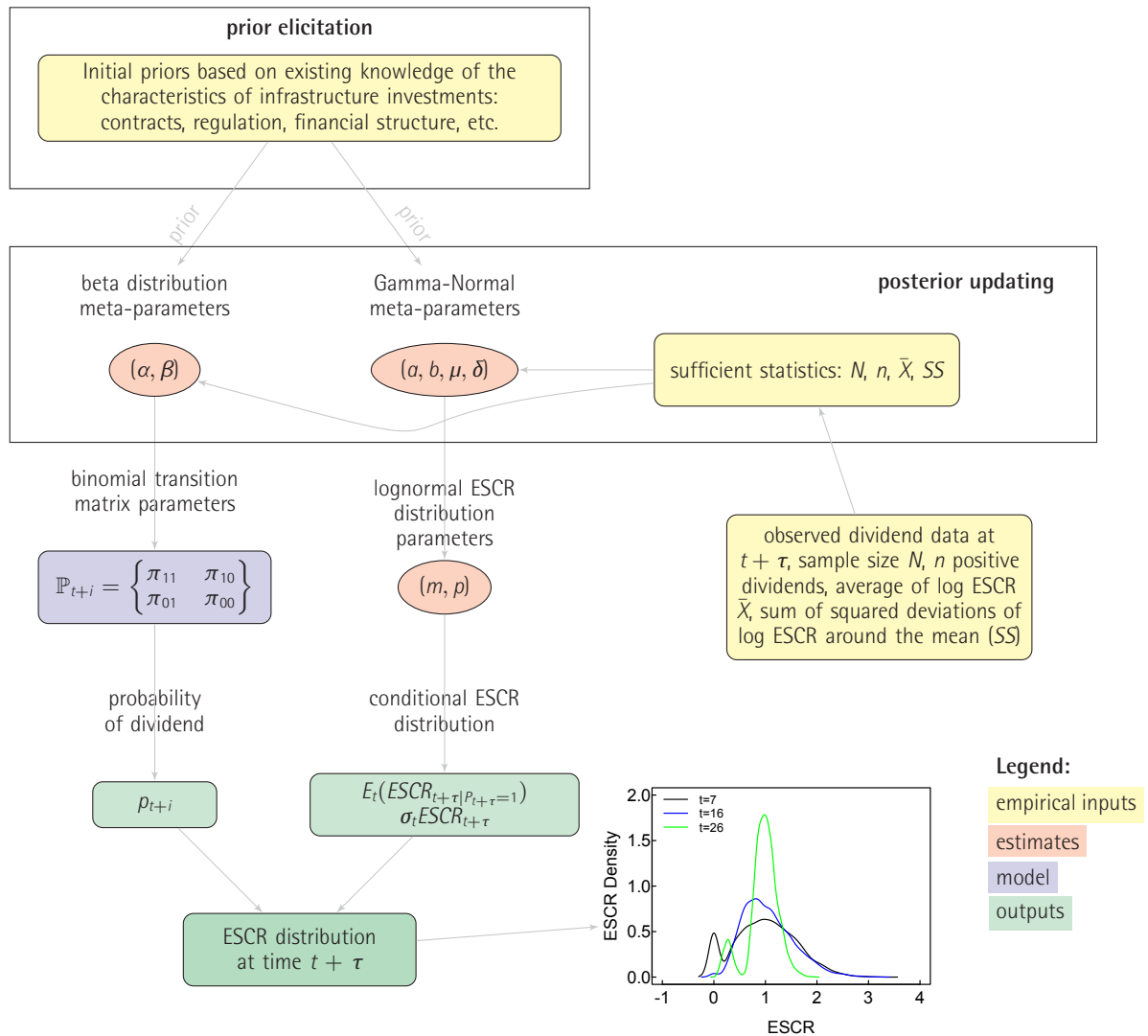
In contrast, utilities or ports and airports may warrant open-ended dividend forecasts and can be expected to exhibit a certain cyclicity driven by the business and regulatory cycles (e.g. 5-yearly determinations of the UK water regulator).

Thus, depending on the underlying business model represented by different types of infrastructure investments, different initial priors may be used and updated using data from the corresponding sub-categories of infrastructure investments.

Note that this use of infrastructure investments *ex ante* characteristics (contracts, financial structure, regulatory schemes, etc.) to categorise investments is itself Bayesian. If large, representative samples of dividend cash flows from privately-held infrastructure investments could be observed today then these characteristics could simply be used a "control variables" of the expected value and volatility of dividends.

2. Expected Dividend Model

Figure 10: How prior knowledge of infrastructure equity investments can be combined with observable data to derive posterior parameters of the distribution of dividends



In the absence of such large samples, categorising different types of infrastructure investments *a priori* implies that each subgroup of private infrastructure equity investment correspond to a different underlying distribution of dividends. Of course such hypotheses can subsequently be tested and potentially falsified once some empirical observations have been made and meta-parameters and parameters have been updated. As the examples above suggest, even very "incorrect" (far from the

true value) and uncertain (large variance) priors can rapidly converge towards the true underlying values.

2.5.2 Data collection requirements

The data required to implement this approach to modelling future dividends in infrastructure investments falls into three categories:

1. Individual investment characteristics needed to identify groups of

2. Expected Dividend Model

infrastructure equity investments that can be expected *a priori* to have different underlying cash flow and dividend distributions between groups, and correspond to reasonably homogenous cash flow processes within groups;

2. Initial investment values and dividend base case forecasts at the time of investing and any subsequent revisions of these forecasts;
3. Actual investment values and realised dividend data needed to update ESCR distribution parameters at each point in the investment's life i.e. N , n , \bar{X} and SS .

The required data is summarised in table 3.

Next, in chapter 3, we discuss how the knowledge of the distribution of $ESCR_{t+\tau}$ combined with actual investment decisions in a given cash flow (dividend) process can be used to derive the implied discount rates of investors and assess the range and trends of expected returns in privately-held infrastructure investments.

2. Expected Dividend Model

Table 3: Data collection requirements to update the distribution of $ESCR_t$

A. Investment decision	
	<i>Calendar and Cash flows</i>
	1. Project dates, life, planned construction start and completion dates
	2. Base case equity investment and dividend cash flows
	<i>Financial structure and covenants</i>
	3. Financial structure of the firm
	4. Base case debt amortisation schedule or calendar
	5. Base case DSCR (in the case of project finance)
	6. Debt covenant (e.g. dividend lockup thresholds/triggers)
	<i>Project characteristics</i>
	7. Foreign exchange risk (y/n)
	8. Country, sector (finite lists to be determined)
	9. Revenue risk profile (merchant, contracted, mixed)
	10. Guarantees (Grantor, ECA*, PRI**, etc.)
	11. ESG* (Equator Principles: are the respected? category A/B/C)
B. After the initial investment	
	<i>One-off events</i>
	1. Actual construction start (date)
	2. Actual construction completion (date)
	<i>Cash flows at time t</i>
	3. Actual capital investment (including any <i>additional equity investment</i> following construction costs overruns or an event default)
	4. Dividend payouts if any

* Export Credit Agency, ** Political Risk Insurance, ★ Environmental & Social Governance

3. Valuation Framework



3. Valuation Framework

In this chapter, we describe a valuation methodology of privately-held infrastructure equity investments for which the distribution of $ESCR_{t+\tau}$ defined in chapter 2 is known, conditional on available information at time t .

In line with academic literature, we argue that the nature of such investments requires estimating a **term structure of discount factors** at different points in their lives that reflects the change in their risk profile. Indeed, *in expectation*, infrastructure investments can exhibit a dynamic risk profile for numerous reasons including the sequential resolution of uncertainty that characterises an investment project with distinct phases (construction, ramp up, operations, maintenance, etc), the frequent de-leveraging of the project company's balance sheet in time (as is often the case in non-recourse project finance vehicles) or the existence of a fixed-term to the investment, e.g. concession contracts, which creates a duration for equity investors but also introduces time-varying duration risk as the investment approaches its term.

As we argued in chapter 1, this term structure of discount factors that investors apply to expected cash flows needs to be endogenously determined because these cash flows always occur in the future.

Of course, once realised returns have been observed, they may be decomposed in terms of exogenous factors such as public market factors or GDP growth. Nevertheless, until observable investments have reached the

end of their lives, the value of realised returns at time t must remain conditional on a discounted stream of future dividends occurring at time $t + \tau$.

We also argue that investors' **discount factors are unobservable** because they are not uniquely determined but are instead in part driven by market dynamics and in part by individual investor preferences. Thus, significant bid/ask spreads can be expected to persist for nevertheless comparable investments. Moreover, this *range* of valuations can also be expected to evolve with the number and type of investors involved in the market for privately-held infrastructure equity at each point in time.

These two important dimensions of the pricing of privately-held and infrequently-traded assets are taken into account in the design of our proposed infrastructure equity valuation methodology. The first point requires the use of a **term structure model** of investors' expected returns in each future period of the investment. The second point can be addressed through a so-called **state-space model**, which relates the unobservable "state" (here, the term structure of discount factors) corresponding to individual investors/transactions to observable quantities such as the invested amount at time t and the distribution of future dividends at time $t + \tau$.

In what follows, we briefly review in section 3.1 the key points of the modern asset pricing literature that justify our approach.

3. Valuation Framework

In section 3.2, we introduce the state-space framework through which we intend to capture the range and the evolution of the pricing of infrastructure equity investments.

State-space models require a so-called *observation or measurement equation* as well as a *state or transition equation*. Section 3.3 describes the design of an observation equation relating observable investment values to discounted future cash flows, while allowing for time-varying discount factors, but also nesting the classic Gordon growth model. Section 3.4 proposes a state equation: an auto-regressive model of the term structure of discount factors, and discusses the estimation of its prior.

Finally, section 3.5 combines the observation and state equations into a dynamic linear model (DLM) that can be implemented as a Kalman Filter to extract the term structure of discount factors implied by actual investments in a given period, and also track its evolution from period to period.

Section 3.6 concludes.

3.1 Modern Asset Pricing Theory

In this section, we briefly discuss the two main theoretical points highlighted above and argue that an adequate valuation framework for privately-held infrastructure assets should a) incorporate the use of time-varying discount factors and b) recognise the absence of unique market prices for such assets and instead aim to capture a

bounded range of values, given observable investment decisions at one point in time.

3.1.1 Time-Varying Discount Factors

Pricing formulas often assume a constant discount rate for all future periods. Private equity managers in particular typically report an internal rate of return or IRR – the constant discount rate that makes the equity investor's Net Present Value (NPV) since the date of investment equal zero – as a proxy for expected or realised returns. During the middle of an investment's life, the IRR is computed as the combination of a stream of realised cash flows and an end-of-period net asset value (NAV), which itself requires discounting remaining future cash flows; that is, reported IRRs typically combine measures of realised and expected returns.

Still, the finance literature has long argued that using such constant and deterministic discount rates can be problematic: the most basic corporate finance textbook examples (see Brealey and Myers, 2014, for example) argue that the use of a single risk-adjusted discount rate for long-lived assets is defective if projects have multiple phases and project risk changes over time as real-options are exercised by asset owners.

Indeed, a constant risk premium does not measure risk properly on a period by period basis, but rather implies that cash flows occurring further in the future are riskier than cash flows occurring earlier (Haley, 1984), which may not be the case, especially given the kind of sequential resolution of uncertainty which

3. Valuation Framework

characterises infrastructure projects. The use of constant discount rates then leads to *biased NPV calculation* (Ben-Horim and Sivakumar, 1988).

Using constant discount rates amounts to assuming that the risk-free rate, asset beta, and market risk premium are all deterministic and constant at all future points in time, while these variables are effectively time-varying and stochastic (that is, conditional on current information, future expected discount rates are stochastic).

While mathematically it is often possible to compute a constant discount rate that yields the same present value than a term structure of discount factors, *this only makes sense if the correct price is already known*. If, instead, the objective is to determine the fair asset value, time-varying discount rates should be used, and the appropriate constant discount rate (IRR) can only be determined afterwards.

Moreover, it is not always possible to determine a unique constant discount rate that reproduces the price obtained using time varying discount rates when the cash flows can switch sign during the life of the investment, as is sometimes the case for infrastructure projects in which equity owners have to add capital following certain types of construction cost overruns or an event of default. In such cases, a unique IRR cannot necessarily be computed.¹⁰

¹⁰ - This is simply a consequence of the fact that the relation between price and IRR is non-linear, and a unique solution is not guaranteed in general.

In addition, even when the correct price is known, the use of a constant discount rate can be inadequate for several reasons:

- In the case of a finite-life investment, using the IRR does not lead to correct **duration** measures;
- Using the IRR to compute the **terminal value** of the investment can yield inaccurate results as the IRR assumes that each cash flow can be reinvested at the same rate;
- The IRRs of individual investments cannot be easily used to estimate performance at the **portfolio** level, as the IRR of a portfolio is not the same as the weighted average IRRs of individual investments;
- IRR-based valuation methodologies cannot be used to identify different **sources of return**, which requires identifying period returns and decomposing them into systematic and idiosyncratic components. In fact, it is possible to build two streams of cash flows with the same IRR but diametrically opposed *betas*.

Ang and Liu (2004) present examples of erroneous valuations resulting from the use of a constant discount rate compared to the use of a term structure of time-varying discount rates and show that this can lead to mis-pricings well over 50%.

Similarly, Phalippou (2008, 2013) provides examples of errors resulting from the use of IRR in evaluating fund performance. Phalippou (2013) shows that due to the use of IRR, the Yale endowment's return since inception on its private equity

3. Valuation Framework

fund stays close to 30% due to a few large capital distributions in early years, and is almost completely insensitive to later performance, making the metric economically meaningless. Phalippou (2008) also highlights that the use of IRRs to measure fund performance, allows fund managers to time their cash flows and boost reported performance measures without increasing investors' effective rate of return.

Thus, using methodologies based on discounting at a constant rate, while common in the corporate sector (Shearer and Forest Jr, 1997; Froot and Stein, 1998; Aguais et al., 2000; Graham and Harvey, 2001), is inadequate for the purposes of long-term investors who need performance measures that can help them make hedging, risk management, and portfolio management decisions.

In appendix 6.5.1, we summarise several examples illustrating the pitfalls of using single discount rates to value long-term investments such as privately-held infrastructure equity.

3.1.2 A Bounded Range of Valuations

Assuming (weak) market efficiency, any traded assets must be uniquely priced i.e. any opportunity to earn a risk-less profit (arbitrage) from a difference in price for the same asset in two markets must quickly disappear. Thus, to the extent that assets are spanned by traded instruments, all investors must hold the same price for the same asset.

With incomplete markets however, since some assets are not fully spanned by traded securities, individual investors can arrive at *different valuations of the same asset*. The proportion of returns that cannot be explained by traded factors may thus lie within a **range of expected returns or discount rates**, determined by individual investors attitudes towards risk, liquidity, inflation, duration etc. For instance, investors' valuation of the potential diversification benefits of investing in unlisted infrastructure is a function of their overall portfolio and of the size of their allocation to infrastructure.

In the case of privately-held infrastructure investments, markets can be considered incomplete for exogenous reasons (there is no easily identifiable portfolio of traded securities which always replicates the payoff of the asset), as well as endogenous reasons (transaction costs are high).

From this perspective, the oft-mentioned illiquidity premium expected by investors in private assets does not have to be unique either. While *relatively* illiquid instruments that are available in sufficient quantity to allow investors to construct arbitrage can yield a unique illiquidity premium, e.g. small cap stocks, unlisted assets that are so infrequently traded that identical assets cannot be bought and sold at the same time, may command a different illiquidity premium for different types of investors. Thus, large bid/ask spreads may persist for investments in private infrastructure projects.

3. Valuation Framework

In the literature, individual investors' attitudes towards risk or their preference for liquidity are usually integrated in subjective valuation models – such as indifference pricing – using an *expected utility framework*.

But this approach requires making numerous assumptions, which are subsequently difficult to calibrate in an operational setting, in particular the ubiquitous risk aversion parameter. Importantly, while internally consistent, the expected utility or indifference pricing framework is limited because it does not take into account the possibility of "market review" (Carr et al., 2001a).

In other words, while the presence of incomplete markets warrants taking subjective valuations into account, the expected utility framework is strictly subjective, whereas the market dynamics of private equity investments call for a more *inter-subjective* understanding of price formation.

For example, if a new type of investor, e.g. less risk averse, enters the private infrastructure equity market, the *range* of observable valuations may change. Likewise, if some investors want to increase their allocations to unlisted assets, given the limited availability stock of investable infrastructure projects at a given point in time, *their* valuations will rise, but not those of others (who may sell). Finally, if infrastructure equity returns can gradually be better hedged using traded assets,

then individual subjective valuations should converge towards a unique pricing measure.

Thus, while the price of a given unlisted infrastructure equity investment is unlikely to be unique and probably lies within a range that at least partly reflects investor's subjective preferences, this range of values is not unlimited and must be *bounded* by the same investor preferences at one point in time.¹¹

In general, it should be possible to observe reasonable limits on the risk/reward ratio that a population of investors *require* from a given privately-held investment, at a given point in time.

To conclude, if privately-held equity investment in infrastructure projects must give rise to range of subjective valuations, only part of which may be explained by traded market factors, it follows that **investors' required returns or discount factors of expected dividends in infrastructure investments are fundamentally unobservable**: they cannot be derived solely from observable transaction prices since they are both a function of the characteristics of the asset, e.g. cash flow volatility, and of individual investor preferences. Each observable transaction thus corresponds to a single pricing equation with two unknowns (project and investor characteristics) and cannot be solved directly.

However, a range of required returns may be implied by observing investment decisions into a well-defined infrastructure cash flow

¹¹ - In the arbitrage bounds literature, Carr et al. (2001b) suggest 'acceptable' investment ranges, while El Karoui and Quenez (1995) discuss super-hedge strategies that correspond to arbitrage bounds on asset prices.

3. Valuation Framework

or dividend process (such as the one presented in chapter 2), using the state-space formulation that we discuss next.

3.2 A State-Space Framework

In this section, we describe the purpose and general setup of the state-space framework.

The objective of state-space models is parameter estimation and inference about *unobservable* variables in dynamic systems, that is, to capture the dynamics of observable data in terms of an unobserved vector – here the term structure of discount factors – known as the state vector of the system (the market). Hence, we must have an *observation or measurement equation* relating observable data to a state vector of discount factors, and a *state or transition equation*, which describes the dynamics of this state, from one observation (transaction) to the next. The combination of the state and observation equations is known as a state-space representation of the system's dynamics.

Our unit of observation is the individual transaction i.e. individual infrastructure equity investments. To each transaction corresponds a given stochastic dividend process characterised by a distribution of $ESCR_{t+\tau}$ with $\tau = 1, \dots, T$ and an investment value P_0 , both of which we assume to be *observable*.

Note, however, that we only "observe" the distribution of $ESCR_{t+\tau|t}$ i.e. given information available at time t . While actual cash flows only occur in the future, their

distribution is considered to be known (as best as possible) when the valuation is done. This is intuitive as long as the parameters of the distribution of $ESCR_{t+\tau|t}$ incorporate all information available at that time t about the dividend process. If new information becomes available at time $t + 1$ so that $ESCR_{t+\tau|t+1}$ is different from $ESCR_{t+\tau|t}$, then the valuation process needs to be redone to incorporate this new information.

Each transaction is the expression of a valuation "state" i.e. a given term structure of discount factors matching the price paid in that transaction (the initial investment) with expected cash flows. As discussed above, this state is *unobservable* because it is partly determined by the investor's subjective preferences and, in the absence of complete markets, cannot be discovered by pricing a portfolio of traded assets that would always replicate the investment's payoff.

Each transaction corresponds to a new state i.e. a new valuation, which may or may not be the same than the previous transaction's. Given a stream of risky future dividends, if the price paid in the current transaction is different from that paid in the previous one, it must be because the valuation state has shifted. The valuation state can change due to a change in investor preferences between the two deals, or due to a change in the consensus risk profile of those kind of investments, e.g. projects with commercial revenues after a recession, or because of a change in the overall market sentiment (the average) valuation.

3. Valuation Framework

Thus, by iterating through transactions, we may derive an implied average valuation state (term structure of discount factors) and its range, bounded by the highest and lowest bidders in the relevant period.

Later, when dividend payments are realised, (conditional) per-period returns can be computed using the discounted sum of remaining cash flows as the end-of-period value (given the implied term structure of discount factors previously derived).

Note however that at each point in time, realised dividends may be sufficiently different from the original base case to lead to a revision of the dividend forecast for a given investment, or that observing several dividend payouts for comparable investments may also lead to a recalibration of the parameters of $ESCR_t$. In both cases, expected dividends need to be re-computed, which will also impact the computation of realised period returns.

Again, as for any other stock, the valuation of privately-held infrastructure equity amounts to deriving the appropriate discount rates for a given estimate of future dividends.

But while this process is *implicit* in the pricing mechanism of public stock markets, in the case of privately-held equity with distant payoffs, we have to derive the relevant inputs *explicitly*, taking into account the characteristics of thinly-traded investments such as infrastructure equity.

3.2.1 State-Space Model Formulation

Typically, the state and observation equations are written, respectively, as follows (Durbin and Koopman, 2012) :

$$\begin{aligned} \theta_k &= H_k \cdot \theta_{k-1} + w_k \\ &\begin{matrix} p \times 1 & p \times p & p \times 1 & p \times 1 \end{matrix} \\ &\text{with } w_k \sim iid \mathcal{N}(0, Q_k) \\ y_k &= F_k \cdot \theta_k + B_k u_k + v_k \\ &\begin{matrix} m \times 1 & m \times p & p \times 1 & m \times pp \times 1 & m \times 1 \end{matrix} \\ &\text{with } v_k \sim iid \mathcal{N}(0, R_k) \\ &\text{so that } E(v_k w'_s) = 0 \forall s, k \end{aligned}$$

where, θ_k is the state vector, u_k is a state-independent input vector, and y_k is the set of observations made at k^{th} transaction. H_k , F_k , B_k , Q_k and R_k are the system matrices of time-variant but non random coefficients for transaction k .

As suggested above, the state equation is the expression of a term structure of discount factors (the vector θ_k) in iteration k expressed as a function of the system's state in the previous transaction $k - 1$ i.e. the vector of discount rates of the previous transaction, plus a state-independent contribution, $B_k u_k$, taking into account the (consensus) views about the change in the volatility of expected cash flows at k^{th} iteration, plus or minus a random factor w_k .

This state "estimation error" can be interpreted as capturing the range of investor valuations of comparable assets in individual transactions, and matrix H_k as capturing the expected return autocorrelation between individual transactions (market sentiment). Matrix $B_k u_k$ captures the effect of change in

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discount rates due to a change in perceived risk profile (the conditional volatility of dividends) between transactions, with u_k denoting the change in risk profile, and B_k denoting the associated repricing of risk.

The observation equation links the observations y in transaction k with the expected cash flows in matrix F_k discounted using vector θ_k , while v_k , the observation error, captures the extent to which expected cash flows are not well documented at the time of valuation. In particular, it can be calibrated with the variance meta-parameter (the measure of ignorance) of the mean $ESCR_{t+\tau}$ discussed in chapter 2.

As we show in section 3.5, such models lend themselves well to numerical techniques such as Kalman filtering, which allow us to estimate the values of the system matrices and those of the unobservable state vector.

Also note that both state and observation equations above are linear in θ . Indeed, dynamic linear models (DLM) assume that the vector of observed quantities is a linear function of the state vector, as well as Gaussian error terms. In section 3.3, we linearise the pricing or discounted dividend equation for this purpose. Note however that these assumptions can be relaxed in some of the extended versions of state-space models (see for example Cappe et al., 2007).¹²

Next, we propose a pricing formula as the observation equation that allows for time-varying discount factors and can be

expressed as function of the dividend base case at time t , the conditional distribution of $ESCR_{t+\tau}$ and a vector of discount factors.

3.3 Observation (Pricing) Equation

In this section, we first formulate a pricing equation based on the standard dividend discount model (3.3.1), we then give it a matrix formulation (3.3.2), and finally its form as the stochastic observation equation of the state-space system.

3.3.1 Pricing equation

Basic setup

Time starts at t_0 when the initial investment is made (sunk) with a value of P_{t_0} . As described on figure 11, cash flows (dividends) are paid at the end of each period (they can be zero cash flows). The investment lasts for T years, after which its terminal value is zero i.e. $P_T = 0$, as is the case in the majority of standalone infrastructure projects. Each equity return R_t occurs between time t and $t + 1$. Returns at time t are computed as:

$$R_t = \left(\frac{P_{t+1} + C_{t+1}}{P_t} \right) - 1$$

and the standard asset pricing formula is written:

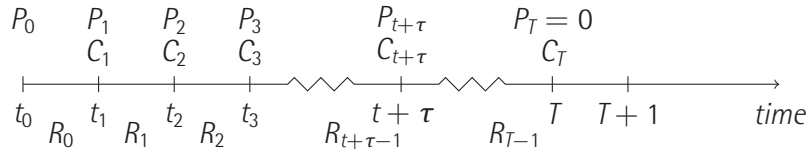
$$P_t = \frac{P_{t+1} + C_{t+1}}{1 + R_t}$$

with P_t the asset price at time t , C_t the cash flow (or dividend) at time t and R_t the expected rate of (gross) return between time t and $t + 1$. Defining r_t as the logarithm of the return relative, so that $1 + R_t = \exp(r_t)$, with \exp the exponential operator, and taking expectations at time t , the asset

¹² - If we do not use the first-order approximation, the problem can be specified as a non-linear state space model, and its parameters can then be estimated using Monte Carlo Markov Chain (MCMC) as in Ang et al. (2013), or sequential Monte Carlo as suggested in Petris et al. (2009).

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Figure 11: Investment and Cash Flow Timeline



price is written:

$$\begin{aligned}
 P_t &= E_t \left(\frac{P_{t+1} + C_{t+1}}{\exp(r_t)} \right) \\
 &= E_t \left(\exp(-r_t)(P_{t+1} + C_{t+1}) \right) \quad (3.1)
 \end{aligned}$$

Following Hughes et al. (2009), and recursively applying equation 3.1 to P_{t+1} , P_{t+2}, \dots , yields the usual transversality condition¹³ (see Samuelson, 1965)

$$\prod_{i=0}^{\infty} \exp(-r_{t+i}) P_{\infty} = 0$$

i.e. in the long run, the discounted value of the asset itself tends to zero. Thus, in the context of an investment in a **finite-life infrastructure project with no terminal value**, we can readily adapt this framework and replace $+\infty$ with T , the *number of periods left* in the project's life, and the pricing formula is written:

$$\begin{aligned}
 P_t &= E_t \left(\sum_{\tau=1}^T \left(\left(\prod_{i=0}^{\tau-1} \exp(-r_{t+i}) \right) C_{t+\tau} \right) \right) \\
 &= \sum_{\tau=1}^T E_t \left(\exp\left(-\sum_{i=0}^{\tau-1} r_{t+i}\right) C_{t+\tau} \right) \\
 &= \sum_{\tau=1}^T E_t(m_{t+\tau} C_{t+\tau}) \quad (3.2)
 \end{aligned}$$

Thus, at time t , the price of the asset is the discounted sum of expected cash flows at time $t + \tau$, with $\tau = \{1, 2, \dots, T\}$ for a project ending in T years. The stochastic

discount factor (SDF), $m_{t+\tau}$, in each period is the product of the exponential of expected log returns (of the exponential of their sum).

Expected Dividends

Since dividends are also stochastic, we can write the expected value at time t of the $t + \tau$ dividend as a function of base case dividends and the expected value of $ESCR_{t+\tau}$ defined in equation (2.1).

$$E_t(C_{t+\tau}) = C_{t+\tau}^0 \times E_t(ESCR_{t+\tau})$$

since the base case at time $t + \tau$ is known at time t .

The asset pricing equation is now written:

$$\begin{aligned}
 P_t &= \sum_{\tau=1}^T E_t(m_{t+\tau} C_{t+\tau}^0 ESCR_{t+\tau}) \\
 &= \sum_{\tau=1}^T C_{t+\tau}^0 E_t(m_{t+\tau} ESCR_{t+\tau})
 \end{aligned}$$

In appendix 6.5.2, we also show that the same pricing formula, when expressed as function of dividend growth rates reduces to a function of the initial base case dividend, base case dividend growth rates, ESCR log returns (as defined in chapter 2) as well as the SDF $m_{t+\tau}$.

We also show in appendix 6.5.3 that when expressed as a function of dividend growth rates, this pricing equation nests

13 - Also known as the "no ponzi game" condition.

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the standard Gordon model (Gordon and Shapiro, 1956) with constant dividend growth and expected returns and can be written as the first term of a two-stage Gordon growth model with constant dividend growth and expected returns (Elton et al., 2009).

Note that while in the rest of this paper we focus on a model of expected dividends in the context of an infrastructure project with finite life, pricing models using stochastic dividend growth rates may be more appropriate for investments in open-ended infrastructure assets such as airports or certain utilities.

Construction period

As discussed earlier, the stream of dividends of a new infrastructure investment may not start immediately, especially during its construction period. We showed in chapter 2 that estimating the components of the state-transition matrices of the dividend process at the time when the first cash flow is expected to occur allows us to adjust expected cash flows for the risk of construction delays.

Thus, at the discounting stage, we can simply discount the zero-dividend construction periods at the risk-free rate i.e. the cost of time (compounded) until the first risky cash flow is discounted at the appropriate rate. With P_{t_0} the price at the beginning of construction period, and P_{t_c} the value at the time of construction

completion, the pricing equation is written:

$$P_{t_c} = \sum_{\tau=1}^T E_t(m_{t+\tau} C_{t_c+\tau}) \quad (3.3)$$

$$P_{t_0} = \exp\left(-\sum_{j=0}^{t_c-t_0-1} r_{t_0+j}^f\right) P_{t_c} \\ = m_{t_0,t_c}^f \sum_{\tau=1}^T C_{t_c+\tau}^0 E_{t_0}(m_{t_c+\tau} ESCR_{t_c+\tau}) \quad (3.4)$$

with r_t^f the risk-free rate at time t and m_{t_0,t_c}^f the additional discount factor capturing the length construction period, during which zero dividend is paid with certainty.

3.3.2 Matrix formulation

The exact relation between prices and discount rates given in equation 3.2 is non-linear and, as discussed above, dynamic linear models require that both the state and observation equations be linear.

While, the state equation discussed next in section 3.4 is linear, the pricing equation needs to be linearised to qualify as an observation equation in a DLM setting.

To linearise the relationship between price and discount rates, we show in appendix 6.5.4 that the first-order Taylor approximation leads to the following relationship expressed in matrix terms:

$$P_{t_c} = \underset{1 \times T}{D'} \cdot \underset{T \times 1}{1} - \underset{1 \times T}{D'} \cdot \underset{T \times T}{A} \cdot (\underset{T \times 1}{R_t^f} + \Lambda_t) \quad (3.5)$$

where $\underset{T \times 1}{1}$ is a vertical vector of ones, $\underset{T \times 1}{D}$ is the vector of expected dividends, $\underset{T \times T}{A}$ is a lower triangular identity matrix, $\underset{T \times 1}{R_t^f}$ is a vector of risk-free rates at T horizons and

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Λ_t is a vector of per-period excess returns $T \times 1$ at the same horizons, and P_{t_c} is the price at the time of construction completion.

Combining all observable variables (initial price, risk free rates, and expected cash flows) on the left hand side, we re-write equation 6.15 as

$$D' \cdot 1 - D' \cdot A \cdot R_t^f - P_{t_c} = D' \cdot A \cdot \Lambda_t \quad (3.6)$$

Equation 3.6 provides a linear relationship between observable quantities and a vector of per-period excess returns Λ_t .

In the presence of a construction period, equation 3.6 can be written in terms of the price at the financial close, P_{t_0} , as:

$$D' \cdot (1 - A \cdot R_t^f) - (1 + \sum_{j=0}^{t_c-t_0-1} r_{t+j}^f) P_{t_0} = D' \cdot A \cdot \Lambda_t$$

$$D' \cdot (1 - A \cdot R_t^f) - (1 + 1'_c \cdot R_{t_0}^f) P_{t_0} = D' \cdot A \cdot \Lambda_t \quad (3.7)$$

where $1'_c = [1 \ \dots \ 1 \ 0 \ \dots \ 0]$ is an indicator vector that contains 1 for periods that fall in the construction period and 0 for periods that do not fall in the construction period, and $R_{t_0}^f$ is the vector of forward risk free rates from the time of financial close, t_0 , until the end of the investment's life at $t_c + T$.

3.3.3 Observation equation

Finally, we can express the linearised version of the pricing equation as the stochastic observation equation of a state-space system described in section 3.2.

Denoting the price at financial close, P_{t_0} , for the k^{th} transaction as P_k , we can denote the

combination of observable variables at the k^{th} transaction y_k as,

$$y_k = D' \cdot 1 - D' \cdot A \cdot R_t^f - (1 - 1'_c \cdot R_{t_0}^f) P_k$$

the unobservable or latent state as,

$$\theta_k = \Lambda_k$$

the observation matrix as,

$$F = D' \cdot A$$

and rewrite equation 3.7 as,

$$y_k = F_k \cdot \theta_k + v_k$$

The addition of an error term $v_k \sim iid \mathcal{N}(0, R_k)$ to equation 3.7 makes it a stochastic equation and encapsulates the uncertainty of our measurement of the data, in particular the precision with which we may know at time t the expected value of future cash flows at time $t + \tau$.

Note, the system dimensions in our model are given by $m = 1$, which corresponds to the number of observation variables (transaction price), and $p = T$, which corresponds to the T unknown discount factors.

3.4 State (Discounting) Equation

In this section, we first describe the discount factor term structure model corresponding to an individual transaction (or valuation) (3.4.1). Next, we give is a matrix formulation (3.4.2) and its form as the stochastic state equation (3.4.3). Finally, we discuss the estimation of its parameters (3.4.4).

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3.4.1 Term structure model

Risk premia and discount factors

The valuation state is a vector of discount factors $(m_{t+\tau})_{\tau=1}^T$ that are applied to each expected cash flow at the relevant horizon, as per equation 3.2.

In order to be in a position to compute these discount factors, we require a model of the term structure of *forward risk premia* (call them $(\lambda_{t+i})_{i=0}^{T-1}$) applied to each future dividend which, added to the relevant vector of risk-free rates (call them $(r_{t+i}^f)_{i=0}^{T-1}$), represent the term structure of expected (or required) period returns of an investor buying this stream of cash flows at time t .

Note that the discount factors $m_{t+\tau}$ simply are the combination of the per-period expected returns $r = r^f + \lambda$ from time t to time $t + \tau$.

In the case of continuously compounded log returns, the multi-period expected return from time t to τ (call it $\mu_{t+\tau}$) is the arithmetic average of period return, so that:

$$\mu_{t+\tau} = \frac{r_t + r_{t+1} + \dots + r_{t+\tau-1}}{\tau} \quad (3.8)$$

and the cumulative return is

$$\tau\mu_{t+\tau} = \sum_{i=0}^{\tau-1} r_{t+i}$$

so that

$$m_{t+\tau} = \exp(-\tau\mu_{t+\tau}) \quad (3.9)$$

Thus, with a model of the term structure of λ_{t+i} and a given term structure of the risk-free rate at similar horizons, we can derive

the per-period expected returns of investors, as well as the discount factors required to value the investment at each point in time.

Term structure of risk premia

As discussed earlier, we argue that the required excess returns applicable to each future period $t + \tau$ is time-varying because of the *expectation* at time t that the equity risk profile of the project will change. We thus posit that to each future cash flow must correspond a *unique forward risk premia*, which embodies both the risk inherent in the cash flow and an investor's preference for this risk.

At time t , for a given forward term structure of continuously compounded risk-free rate $R_t^f = (r_{t+i}^f)_{i=0}^{T-1}$, investors require a series of per-period expected returns $R_t = R^f + \Lambda_t = (r_{t+i}^f + \lambda_{t+i})_{i=0}^{T-1}$ with Λ_t the forward curve of risk premia.

A handful of academic papers have explored analytical and computational solutions to discount cash flows with time-varying expected returns. For example, Ang and Liu (2004) propose a cash flow discounting framework for stocks with time-varying expected returns. However, their discount factors are exogenously determined by the state of markets, the economy, etc. and not (endogenously) determined by actual investment values (prices).

Such factor models are useful to determine discount rates when factors and factor loadings are observable. In our case however, dividend payouts lie far in the future, and correlations of infrastructure

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equity returns with 'remunerated' factors are unknown. Hence, as argued above, we must rely on a model of the **endogenous** determination of the forward curve of risk premia.

To this effect, we make two hypotheses: first, the term structure of expected period returns is explicitly autoregressive with one-lag (AR(1)) implying a degree of **persistence** (the opposite of randomness) between expected returns in time; second, we posit that – for a given set of investor preferences – *conditional dividend volatility explains required returns* i.e. investors' required forward premia are a function of the (conditional) volatility of future cash flows.

Of course, assuming no-arbitrage, returns on any asset should be determined by the *covariance* of the asset's returns with the returns of one or more risk factors. For example, with a single factor model, we can write the value of the excess return (or required risk premia) λ_t^i of asset i at time t as,

$$\begin{aligned}\lambda_t^i &= \beta_t^M \lambda_t^M \\ \lambda_t^i &= \frac{\text{Cov}(\lambda_t^i, \lambda_t^M)}{\text{Var}(\lambda_t^M)} \lambda_t^M \\ \lambda_t^i &= \rho_t \frac{\lambda_t^M}{\sigma_t^M} \sigma_t^i\end{aligned}$$

where, λ_t^M is the excess return on the market at time t , σ_t^i and σ_t^M are the volatilities of excess asset and market returns at time t , and $\beta_t^M = \frac{\text{Cov}(\lambda_t^i, \lambda_t^M)}{\text{Var}(\lambda_t^M)}$ is the asset's market beta.

Note that this formulation could be extended to a vector of multiple factors

and factor loadings. However, as we argued above, we cannot project factor values in the future and, in a model of endogenously determined risk premia, we can treat all exogenous factors as a single quantity.

Next, defining $\gamma_t = \rho_t \frac{\lambda_t^M}{\sigma_t^M}$ the above equation can be re-written as

$$\lambda_t^i = \gamma_t \sigma_t^i \quad (3.10)$$

Hence, the asset beta(s) (factor loadings) while unknown are implicitly taken into account by empirically estimating parameter γ_t given the volatility σ_t^i which is asset-specific and knowable.

Next, a factor model may not hold exactly in incomplete markets, and the required return may be higher than what is implied by factor loadings. Hence, we add an *alpha* constant to the state equation:

$$\lambda_t^i = \alpha_t + \gamma_t \sigma_t^i \quad (3.11)$$

where α_t is the return in excess to the factor-implied return.

If this excess return is persistent, i.e. $\alpha_t = \phi \alpha_{t-1}$, then we can write

$$\begin{aligned}\lambda_t^i &= \phi \alpha_{t-1} + \gamma_t \sigma_t^i \\ &= \phi(\lambda_{t-1} - \gamma_{t-1} \sigma_{t-1}^i) + \gamma_t \sigma_t^i \\ &= \phi \lambda_{t-1} + \gamma_t \sigma_t^i - \phi \gamma_{t-1} \sigma_{t-1}^i \\ &= \phi \lambda_{t-1} + \gamma_t (\sigma_t^i - \sigma_{t-1}^i) \\ &\quad + \sigma_{t-1}^i (\gamma_t - \phi \gamma_{t-1}) \\ &= \phi \lambda_{t-1} + \gamma_t \dot{\sigma}_t^i + \sigma_{t-1}^i (\gamma_t - \phi \gamma_{t-1}),\end{aligned}$$

where $\dot{\sigma}^i = \sigma_t^i - \sigma_{t-1}^i$ is the change in return volatility. If γ_t , which is essentially the price of risk, changes slowly in time compared

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to the asset return volatility, then the contribution of the last term in the above equation is relatively insignificant ($\gamma_t - \phi\gamma_{t-1} \approx 0$), and we ignore it in computing the term structure of an individual deal, and incorporate this approximation error later in the evolution of term structure between deals (equation 3.18). Thus, the excess return can be written as

$$\lambda_t^i = \phi\lambda_{t-1}^i + \gamma_t\sigma_t^i \quad (3.12)$$

The difference equation 3.12 implies that the require risk premia in any period of the investment is determined by a persistence component (the previous period's expected return, $\phi\lambda_{t-1}^i$) and the change in asset risk between the two periods, times a risk premium γ_t . The persistence coefficient (ϕ) and risk aversion parameters (γ_t) are unknown, and their values need to be estimated empirically. We return to this in section 3.5.3.

3.4.2 Matrix formulation

We define a vector of period discount rates that consists of discount rates for all future cash flows as:

$$\Lambda_t = \begin{bmatrix} \lambda_t \\ \lambda_{t+1} \\ \vdots \\ \lambda_{t+T-1} \end{bmatrix}. \quad (3.13)$$

Λ_t denotes the term structure of discount rates from t to T , and satisfies the following equation :

$$\Lambda_t = \Phi\Lambda_{t-1} + \Gamma_t\dot{\Sigma}_t \quad (3.14)$$

where

$$\Phi = \begin{bmatrix} \phi & 0 & \cdots & 0 \\ 0 & \phi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi \end{bmatrix} \quad (3.15)$$

$$\Gamma_t = \begin{bmatrix} \gamma_t & 0 & \cdots & 0 \\ 0 & \gamma_{t+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_{t+T-1} \end{bmatrix} \quad (3.16)$$

and

$$\dot{\Sigma}_t = \begin{bmatrix} \dot{\sigma}_{t+1} \\ \dot{\sigma}_{t+2} \\ \vdots \\ \dot{\sigma}_{t+T} \end{bmatrix} \quad (3.17)$$

Equation 3.14 can be used to express a relationship between the term structures of expected returns in two consecutive transactions, k and $k + 1$.

Say (for now) that investor preferences do not change between the two transactions, Γ_{k+1} would be equal to Γ_k . In this case, the term structure of discount rates for the $(k + 1)^{th}$ investment is simply determined by the persistence component of the term structure of the k^{th} deal, plus a potential contribution of the change in the risk profile ($\dot{\Sigma}_{k+1}$) between the two deals:

$$\Lambda_{k+1} = \Phi^{\delta t}\Lambda_k + \Gamma_{k+1}\dot{\Sigma}_{k+1}$$

where Λ_{k+1} is the term structure of the $(k + 1)^{th}$ investment, Λ_k is the term structure of

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the k^{th} investment, and $\Gamma_{k+1}\dot{\Sigma}_{k+1}$ reflects the premium arising from a change in the consensus view about the volatility of expected cash flows at the time of transaction $k + 1$.

3.4.3 State equation

Since investor preferences can indeed vary from one transaction to another, as different investors are involved in different transactions, the term structure of the next investment given the term structure of the current one may have a random term, and can be written more generally as,

$$\Lambda_k = \Phi^{\delta t} \Lambda_{k-1} + \Gamma_k \dot{\Sigma}_k + w_k \quad (3.18)$$

where $w_k \sim N(0, Q_k)$ captures the random variation in term structure between successive deals due to variation in investor preferences, as well as any approximation errors in the term structure model.

This expression is also that of the state or transition equation discussed above. Denoting

$$\begin{aligned} \Lambda_k &= \theta_k \\ T \times 1 & \quad T \times 1 \\ \Phi^{\delta t} &= H_k \\ T \times T & \quad T \times T \\ \Gamma_k &= B_k \\ T \times T & \quad T \times T \\ \dot{\Sigma}_k &= u_k \\ T \times 1 & \quad T \times 1 \end{aligned}$$

Equation 3.18 can be re-written as:

$$\theta_k = H_k \theta_{k-1} + B_k u_k + w_k$$

3.4.4 Empirical derivation

The empirical estimation of expected returns would require complete time series of observed equity prices, but as we argue in chapter 1, this is unavailable to us.

Instead, to obtain numerical estimates of each period's expected excess returns, we can use observable expected dividends and dividend volatilities: we first provide an analytical expression of each period's price in terms of the dividend distribution, and then use this time series of prices to infer the term structure of expected returns.

Assuming that the price at maturity, P_T , is zero, we can write the previous period's price, P_{T-1} , by discounting the aggregate cash flow at time T by a discount rate given by equation 3.12. Following this procedure, described in appendix 6.5.5, we continue to move backward from period to period, and write prices in terms of future cash flow distributions, eliminating expected returns and return volatilities.

These steps involve both the volatilities of returns and the volatilities of cash flows. To distinguish between the two, we denote them by σ_{λ_t} and σ_{C_t} , respectively. Moreover, for notational convenience, we drop the superscript i as all variables correspond to the same asset, and denote expected cash flows simply as C_t .

We show that given expected dividends and dividend volatilities until the maturity date T , prices for all times can be computed in this recursive manner according to:

$$\begin{aligned} P_t &= \sum_{\tau=1}^T \left[\frac{C_{t+\tau} - \frac{1}{\phi^\tau} (\sum_{i=1}^{\tau} \gamma_{t+i}) \sigma_{C_{t+\tau}}}{\prod_{j=0}^{\tau-1} (1 + r_{t+j}^f)} \right] \\ P_{t_0} &= \exp \left(- \sum_{j=0}^{t-t_0-1} r_{t_0+j}^f \right) P_t \end{aligned} \quad (3.19)$$

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where P_t is the price immediately before the first cash flow, and P_{t_0} is the price at the time of investment if there is a construction period. This equation does not contain expected excess returns, $(\lambda_{t+i})_{i=0}^{T-1}$, and return volatilities, $(\sigma_{t+i})_{i=0}^{T-1}$, as those have been expressed in terms of cash flow characteristics.

Using the prices determined as a function of cash flow characteristics and investor preferences in equation 6.22, prior estimate for the expected excess return at time t and their volatilities can be derived as follows

$$\lambda_t = \frac{P_{t+1} + C_{t+1}}{P_t} - r_t^f - 1 \quad (3.20)$$

$$\sigma_{\lambda_{t-1}} = \sigma_{\lambda_t} - \frac{\lambda_t - \phi\lambda_{t-1}}{\gamma_t}, \quad (3.21)$$

starting with $\sigma_T = 0$.

Equations 3.20 and 3.21 relates excess expected returns to cash flow characteristics, which are assumed to be known as they can be estimated from observed equity cash flow data. Using this formulation, in the next section we discuss the estimation of ϕ and γ with maximum likelihood estimators in the context of the state-space model.

3.5 Kalman Filtering

In this section, we show how the observation (pricing) and state (discounting) equations combine into a state-space model that allows us to filter the vector of implied risk premia Λ_t for a series of investments in a given dividend process. We first summarise the full state-space model (3.5.1) before discussing the filtering process (3.5.2), the

estimation of the model parameters (3.5.3), and describing the model outputs (3.5.4).

3.5.1 State-space model

Given the expression of the observation and state equations above, the state-space model is still written:

$$y_{k+1} = F_k \theta_k + v_k \quad (3.22)$$

$$\theta_k = H_k \theta_{k-1} + B_k u_k + w_k \quad (3.23)$$

The two error terms $v_k \sim N(0, R_k)$ and $w_k \sim N(0, Q_k)$ represent measurement and state transition noise, and are assumed to be independent from each other.

The ratio $\frac{Q_k}{R_k}$ is called the signal-to-noise ratio, and determines how much more information is obtained about the latent state from each new observation. The higher (smaller) the ratio, the more (less) informative each new observation is. This is intuitive since very noisy observations (very uncertain estimates of expected dividends) would drown any signal coming from prices and prevent updating the prior state with any relevant new information. In other words, if we cannot forecast cash flows with any accuracy then no new information can be obtained from observing new transactions.

In this setup, the observation equation (equation 3.22) relates the observable quantity to the unobservable (or latent) state, and the state transition equation (equation 3.23) determines the evolution of the state variable. Thus, given observable quantities at the time of investment (price, expected cash flows, risk free rates), one can estimate a term structure of discount rates

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using the observation equation, and use the state transition equation to predict the term structure and price of the next transaction. Once the next transaction is observed, the difference between the predicted price and the actual price can be used to update our knowledge of the true state of the system and its evolution in time.

This process can be repeated continuously over transactions with a given equity risk profile for a given reporting period to fine tune the estimate of discount rate term structure for each family of projects.

This model can also be used to track both the average level of discount rates in the market (the expected returns of a representative investor) and bounds around this term structure created by the heterogeneity of investor valuations. Indeed, the model treats individual prices as noise around a true (average) level, and the prices estimated by the model would reflect the average level of prices in the market. The variance of the observation residuals (difference between estimated and observed prices) captures the spread around these prices, which may result from the heterogeneity in investor preferences or their beliefs regarding project's risk profile.

The variance matrix of the state transition equation captures the uncertainty associated with the evolution of the term structure from deal to deal, and determines the accuracy with which the prices and discount rates for future transactions can be predicted.

That is, the higher the forecasting variance, the more uncertain one is about the term structure of expected returns in the next transaction, even if the current term structure was known with certainty. This uncertainty arises primarily due to the heterogeneity of investors' preferences and beliefs. If all investors had identical preferences and beliefs about cash flow distributions, the term structure of the representative investor would presumably not change much for identical and consecutive transactions.

3.5.2 Filtering and updating

The main advantage of using a dynamic linear model is that the distributions of discount rates and forecasted prices can be computed analytically using Kalman filtering, which is far more computationally efficient than numerical simulation based techniques such as Markov Chain Monte Carlo (MCMC).

The first phase of the Kalman filter consists of predicting the state at the next iteration (here, a transaction) before observing the data (transaction price) but taking into account any change in the state control variables (here, the conditional volatility). A new state and a new covariance error matrix are thus computed. Next, a noisy measurement is made (a transaction is observed) and the update phase takes place, generating a new estimate of the state of the system.

3. Valuation Framework

As determined in section 3.4, the state of the system is:

$$\theta_k = H_k \theta_{k-1} + B_k u_k + w_k$$

where

- θ_k is a T-dimensional state vector of period log risk premia $(\lambda_{t+i})_{i=0}^{T-1}$
- H_k is the state transition model (a $T \times T$ matrix) at the k^{th} iteration
- B_k is the control input model (a $T \times T$ matrix) applied to a $T \times 1$ vector u_k of the change of volatility of future cash flows given information at time k
- w_k is the process noise, typically assumed to follow $N(0, Q_k)$, with Q_k the covariance matrix.

Thus, the state transition process relates the vector of required risk premia of the previously observed transaction to the current state, implying some degree of consistency between investors' valuations of contracts yielding cash flows drawn from the same distribution, and a transaction/investor specific "innovation" driven by the process noise Q_k .

The control input model takes into account the (consensus) views about the change in the volatility of expected cash flows at the time of the observed transaction. For example, starting from transaction k onward and following a persistent change in economic conditions, the volatility of cash flows in real toll roads may be considered higher (or lower) for the foreseeable future.

At time k a measurement y_k is made (a transaction price is observed) so that

$$y_k = F_k \theta_k + v_k$$

where

- y_k is an 1-dimensional observation
- F_k is the observation model (the pricing equation), an $1 \times T$ matrix, which maps the current state into each observation
- v_k is the observation noise, assumed to be $N(0, R_k)$ with covariance R_k

Here, the observation equation simply expresses observable information in terms of the (assumed known, but noisy) expected cash flows of the contract H_k at the time of the transaction, times a discount factor, which is the state variable. v_k is the noise created by our limited ability to know the value of expected cash flows with precision at the time of the transaction.

The recursive nature of the model implies that knowledge of the previous state is sufficient to compute the estimate of the current state. Hence, state estimation is done in the following order:

1. *Prediction stage*: we first use the state estimate from the previous step (transaction) to produce an estimate of the excess return term structure in the current step. This estimate is a *prior* in the sense that no observation has been made yet, and denoted by the "minus" exponent. That is,

$$\hat{\theta}_k^- = H_k \hat{\theta}_{k-1}^- + B_k u_k, \text{ the prior state estimate}$$

$$P_k^- = H_k P_{k-1}^- H_k^T + Q_k, \text{ the prior covariance}$$

3. Valuation Framework

2. *Updating stage*: This prior is combined with the current observation to refine the state estimate and estimate a *posterior* state denoted by a "plus" exponent:

$$b = y_k - F_k \hat{\theta}_k^-$$

the innovation or measurement residual

$$S_k = F_k P_k^- F_k^T + R_k$$

the innovation or residual covariance

$$K_k = P_k^- F_k^T S_k^{-1}$$

the Kalman gain, is an $m \times n$ matrix

$$\hat{\theta}_k^+ = \hat{\theta}_k^- + b K_k$$

the posterior state estimate

$$P_k^+ = (I - K_k F_k) P_k^-$$

the posterior covariance

This process is summarised in figure 12. A proof of these (standard) results can be found in, for example, du Plessis (1967).

3.5.3 Parameter estimation

The persistence and risk aversion parameters, ϕ and γ_{t_i} , described in section 3.4.1, need to be estimated empirically to derive a term structure θ_0 .

For this purpose, the Kalman filter can be used to implement Maximum Likelihood Estimation (MLE). MLE seeks to find unknown parameters such that the likelihood that of the variable of interest following a pre-specified distribution is maximised.¹⁴

In our case, the variable of interest is the transaction price (the initial investment), which is observable and its distribution, which can be inferred from the data. MLE

estimates the values of parameters ϕ and $(\gamma_{t+i})_{i=1}^T$, such that the model implied transaction prices are most likely to come from the same distribution as the observed transaction prices.

A more detailed presentation of the MLE procedure is presented in appendix 6.5.6.

Once values for ϕ and γ are determined, they can be used in equation 3.20 to compute numerical values for expected excess returns.

Numerically, we start with an initial guess for ϕ ($= 1$) and γ ($= 1$)¹⁵, compute the initial transaction price (P_{t_0}) and the corresponding term structure of excess returns using equations 3.19 and 3.20, and forecast the distribution of prices for the next transactions using the Kalman filter described above.

We use a first round of observed transactions (say, 20) to iterate over the initial guess, and compare the distribution of forecasted prices with the distribution of observed prices, for each iteration.

Kalman filtering allows the values of ϕ and γ to be estimated, as well as their variance and that of the implied term structure of expected returns.

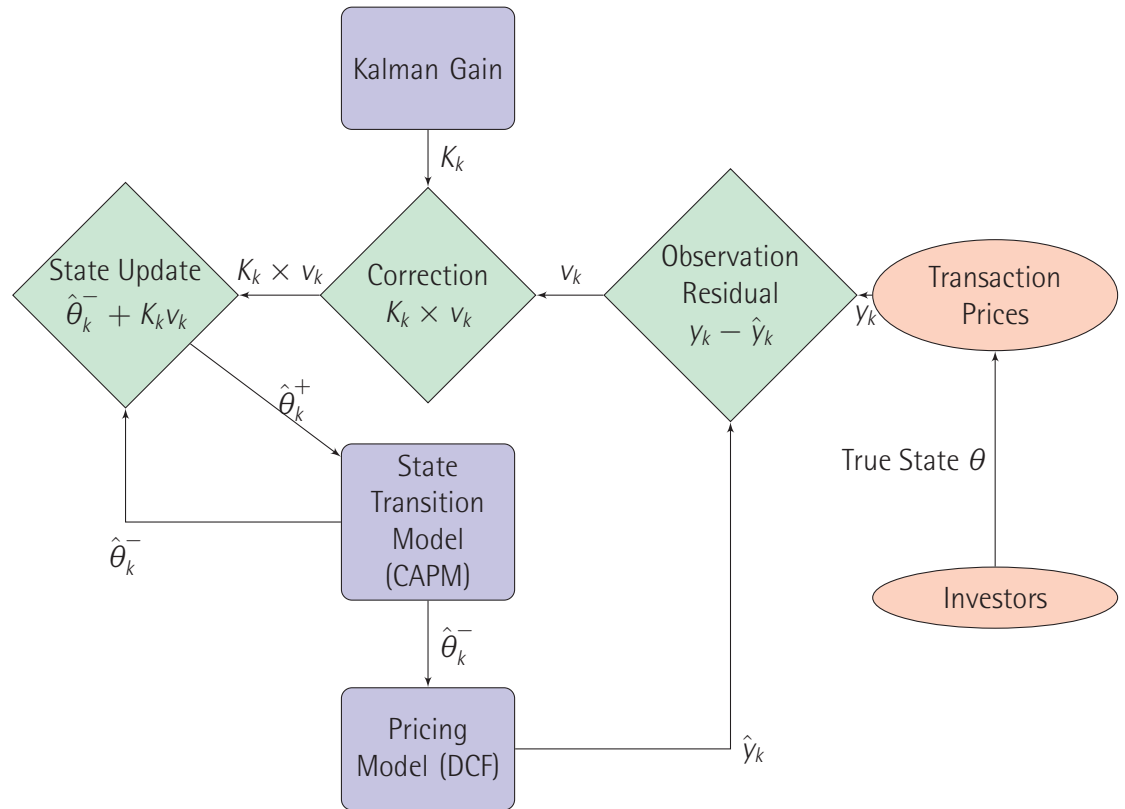
In addition, the term structure parameters and the signal-to-noise ratio may be time-varying as market participants may change, or they may change their preferences and prices may become more or less noisy –

14 - If a distribution for prices cannot be assumed, one can use Generalised Method of Moments to estimate these parameters.

15 - Several initial values for these parameters were tested. The impact on the results (appendix 6.5.7) is very small.

3. Valuation Framework

Figure 12: State-space predicting and updating procedure with Kalman filtering



i.e. the dispersion of individual investors' preferences may vary in time.

Thus all the model's parameters are estimated on a rolling basis as new observations are made, e.g. every 20 observations. This can be achieved with a *rolling* MLE. That is, instead of computing the parameters just once using some initial observed prices, we recompute the desired parameters on a continuous basis, thus capturing the evolution of the market at regular intervals.

The filtering process described in figure 12 follows the following steps:

1. Use an initial guess for the parameters;

2. Compute the transaction price, P_{t_0} , (using equation 3.19) and the corresponding term structure ($\hat{\theta}_k^-$ on figure 12) for the selected values of the parameters (using equation 3.20);
3. Forecast the distributions of prices for the next n transactions using the state space model (\hat{y}_k on figure 12);
4. Obtain the observed prices for n transactions (y_k on figure 12);
5. Estimate the accuracy of forecasted prices ($y_k - \hat{y}_k$ on figure 12) given observed transaction prices;
6. Compute the Kalman gain K_k and update the value of the state estimate to $\hat{\theta}_k^+$ and its variance;

3. Valuation Framework

7. Use this new estimate of the term structure to compute the implied average price and its range;
8. These posterior values become the priors of the next iteration until the accuracy of forecasted prices is maximised (until the forecasting error is minimised).

Thus, thanks to the state-space model setup and its Kalman filter implementation, MLE allows us to compute asset prices for different possible values of the term structure parameters, and pick those values that minimise the forecasting error of the model.

3.5.4 Model outputs

Given available observations, the Kalman filter provides the (statistically) optimal estimates of expected excess returns, $(\lambda_{t+i})_{i=0}^{T-1}$ and the average transaction price, P_{t_0} , along with the variances of these estimates, for a group of observations, e.g. all the transactions observed in a given semi-annual or annual reporting period.

The forward excess returns are related to the transaction prices according to equation 3.7, which is a linearised version of equation 3.4.

Filtered transaction prices determine the price a representative (average) investor would have paid for an equity investment with the given risk profile (future dividend distribution), and the forward excess returns measure the single period expected risk premia that the average investor expects (or requires) during the life of the investment. The standard deviations of filtered prices and forward excess returns measure the

deviation of observed individual investors' prices and required returns from that of a representative investor, and are a proxy for investor heterogeneity.

With this model, we can thus measure the implied valuation (term structure of discount factors) and range of valuations for a given type of infrastructure equity investment during a given reporting period.

We can also track the evolution of this implied valuation and of its range from period to period i.e. track the market dynamics of privately-held and infrequently traded investments that do not command a unique price.

The evolution from period to period of the implied bounds or range of investors' valuations in particular, can provide a powerful measure of the market dynamics of private markets. We provide an illustrative implementation of the model in Chapter 4.

3.6 Conclusion

We have argued that the non-observability of investor's expected returns renders the estimation of *latent* variables a signal-extraction problem.

Since discount factors for privately-held infrastructure equity are not observable, we have specified a model combining a state vector (the term structure of discount rates) that follows an AR(1) process, and an observation equation that relates the state vector to observed prices. In order to retain the simplicity of linear models,

3. Valuation Framework

we use a first-order Taylor approximation to linearise the observation equation. The resulting system of observation and state space equations is estimated using Kalman filtering. This allows us to filter the term structure of implied period excess returns and to compute the term structure of discount rates from observed prices.

This approach thus extends the existing literature in two ways: 1) it introduces a procedure to extract multiple latent factors in a linear framework and 2) it explicitly addresses and aims to capture the heterogeneity of investor preferences (market incompleteness).

Next, in chapter 4, we provide an illustration of this methodology for a generic, time bound, equity investment in an infrastructure project, giving rise to a stream of expected cash flows

4. Illustration



4. Illustration

In this section, we present an implementation of our model for a generic infrastructure equity investment with an assumed dividend distribution, which could have been obtained from the cash flow model described in chapter 2.

We show how the expected returns and the bounds on expected returns can be obtained for this dividend process, with a few observed transaction prices. Next, we show how this can be done continuously in time to track market dynamics i.e. the level and bounds on market prices and the implied IRRs (given the correctly computed price).

4.1 Dividend Distribution

We consider a simple infrastructure project with a 3-year construction period, a 25-year debt repaying period and a 3-year tail (the period during which all debt is expected to be repaid). The base case dividend stream used for this example is a modified version of that of an actual project financed in Europe in 2006.

The expected dividend and dividend volatility are shown in figure 13. We assume that the annualised dividend volatility as a percentage of the base case dividend decreases continuously from 70% immediately after the construction period to about 40% at the expiration of the loan, and then goes up to about 50% during the loan's tail. The end of the equity stream is more considered more volatile (at $t = 0$) because any event of default would lead senior lenders to restructure the senior debt

in the tail, which would concentrate equity losses. Refurbishment (handover) costs may also be higher than expected and create equity losses at the end of this (finite) investment's life. Conversely, expected revenues may be higher than expected and, in the long-run, this effect would also be magnified in the tail when equity holders no longer have any obligations vis-a-vis their creditors.

The assumed probabilities of being in the payment and non-payment states defined in chapter 2 are shown in figure 15. Assumed transition probabilities between payment and non-payment states in each period are shown in figure 14.

The (conditional) probability of not getting paid is low and decreases with time even though we assume that the probability of staying in the non-payment state increases in time conditional on having been in the non-payment state in the previous period i.e. if issues preventing positive distributions are not resolved, they becomes less likely to be resolved at the next period as the life of the project unfolds.

Note that while the dividend profile is inspired by an actual project, these volatilities and probabilities are assumed and purely for illustration purposes.

4.2 Return Profile

The implied term structure of risk premia computed using the assumed inputs are shown in figure 16 and the expected cash yield and duration in figure 19.

4. Illustration

Figure 13: Assumed dividend distribution: The left panel shows the expected dividends, and the right panel shows the dividend volatility as a percentage of base case dividend in each period.

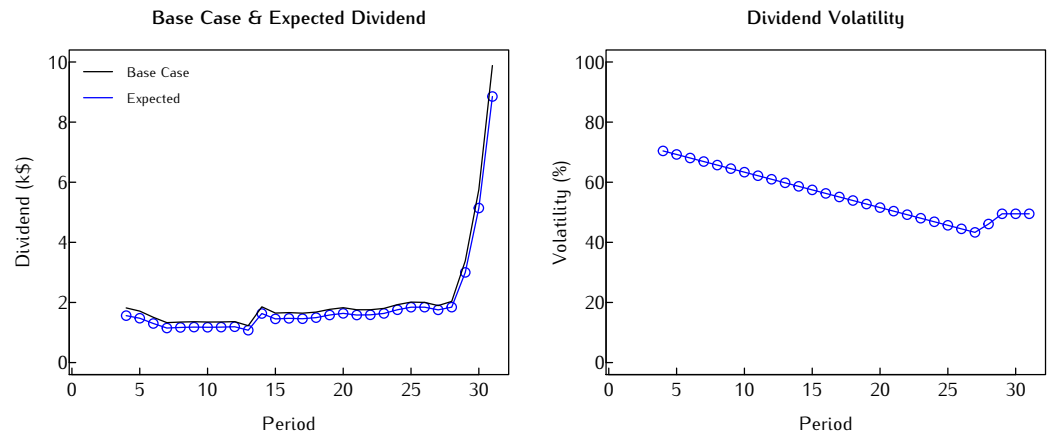
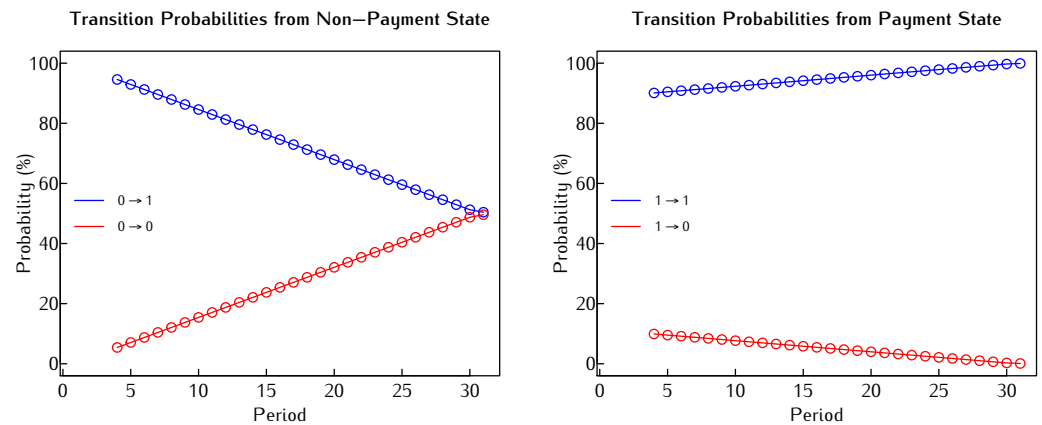


Figure 14: Assumed dividend distribution: The left panel shows the The expected dividend and dividend volatility is shown in figure 13., and the right panel shows the transition probabilities from payment state to both payment and non-payment states. The non-payment state is denoted by 0 and the payment state is denoted by 1.



We start with the assumed dividend characteristics depicted in figures 13 to 14, and assume twenty observed prices such that the IRR is between 12-13%, and then filter over these observations one by one to form expectations regarding future expected returns.

In the left panel, figure 16 shows the assumed risk free forward curve, $r_{t+\tau}^f$, which

roughly matches the treasury forward rate curve.

The right panel shows the filtered excess returns, $\lambda_{t+\tau}$, obtained after filtering over 20 transactions. Excess forward returns initially decrease as the dividend volatility decreases, but then increase near the loan's tail as dividend volatility rises. Changes in expected dividend payouts also impact

4. Illustration

Figure 15: Assumed dividend distribution: The left panel shows the probabilities of being in payment non-payment states, and the right panel shows the expected loss in each period as a percentage of that period's base case dividend.

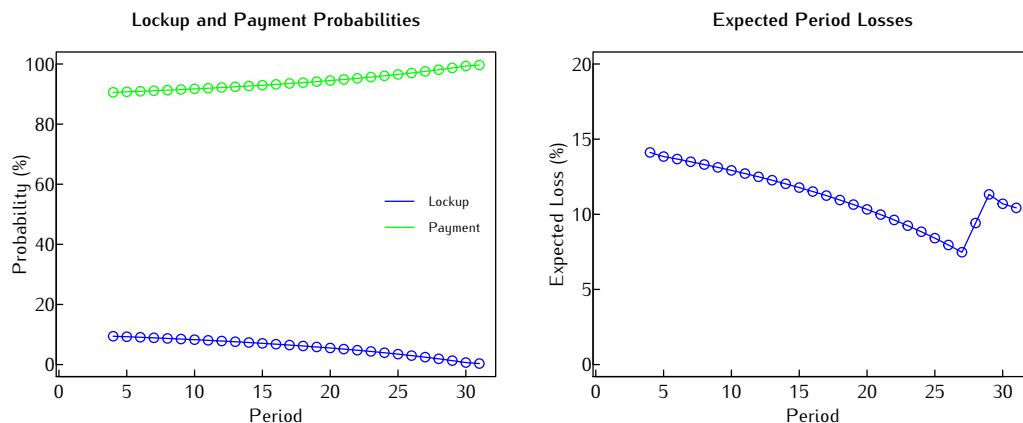
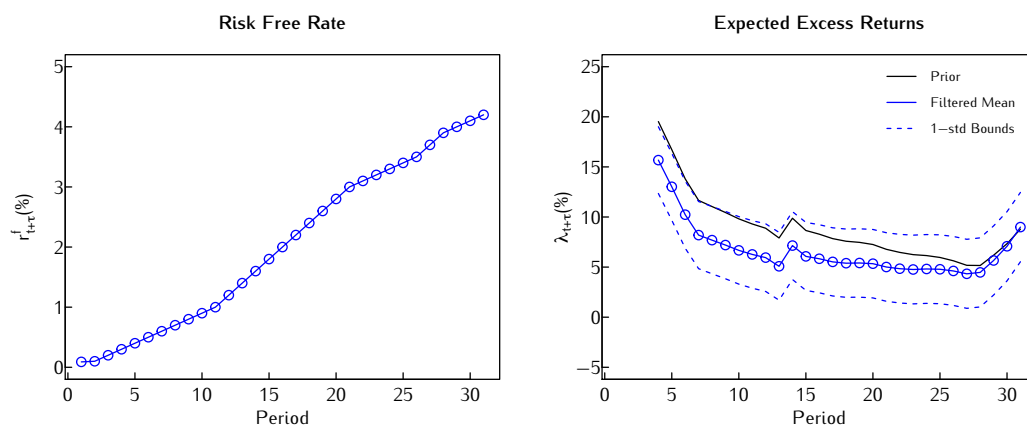


Figure 16: The risk-free forward curve, $r_{t+\tau}^f$, used in the example is shown in the left panel, and filtered estimates of expected excess returns, $\lambda_{t+\tau}$, after observing 20 transactions are shown in the right panel.



expected returns and explain, for example, the *kink* in the term structure in year 14.

The term structures of risk free rates and excess returns are combined to compute per-period expected rates of returns ($r_{t+\tau}$) and multi-period (average) rates of return ($\mu_{t+\tau}$), as shown on figure 17 on the left and right panels respectively.

Single period rates are obtained by adding the single period risk-free rates (forward risk

free rates) to the expected excess returns, and multi period expected returns are obtained from the single period expected returns using equation 3.8.

Next, figure 18 shows the stochastic discount factor, $m_{t+\tau}$ defined in equation 3.9, and the expected (at t_0) evolution of price during the investment's life.

4. Illustration

Figure 17: Filtered estimates of the term structure of single period returns, $r_{t+\tau}$, are shown in the left panel, and multi-period discount rates, $\mu_{t+\tau}$, after 20 transactions are shown in the right panel. These expected returns are based on information available at time t_0 , i.e. the observed 20 transaction prices and dividend distribution.

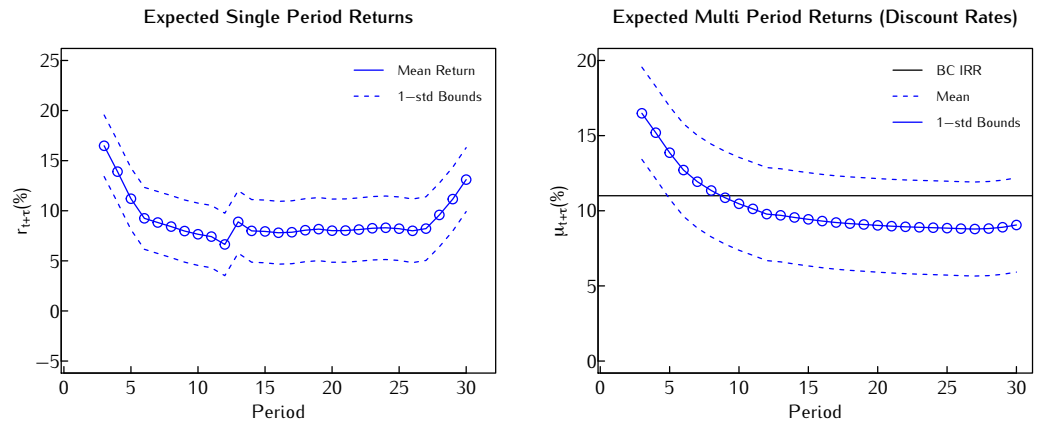
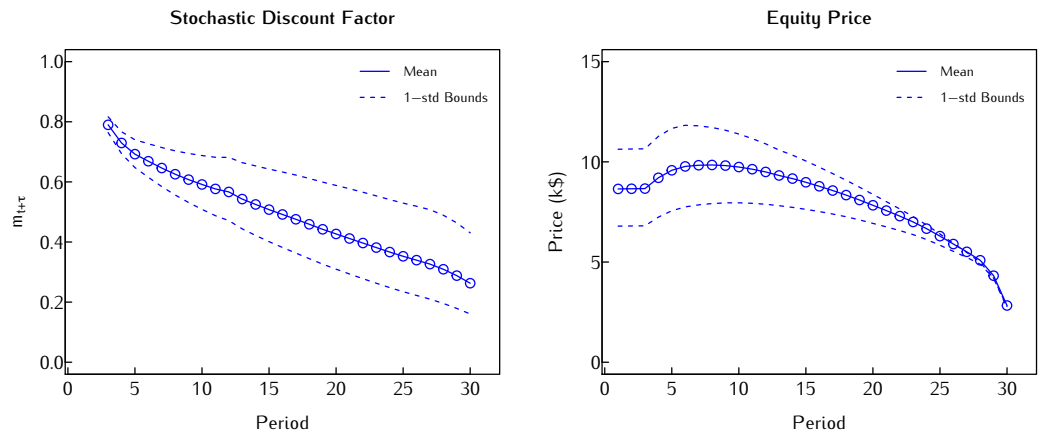


Figure 18: Stochastic discount factor, $m_{t+\tau}$, is shown in the left panel, and the expected evolution of equity price, P_{t_0} , is shown in the right panel. The price curve shows how the price would evolve based on current expectations, i.e. if no new information becomes available in time, and investors continue to use the ex-ante discount factors to price the project in future periods.



The stochastic discount factor magnifies the effect of variations in longer term discount rates, as they are multiplied by length of the horizon to obtain the discount factors. That is, for a given range of discount rates, the discount factors for the longer horizon cash flows exhibit greater deviation than the shorter horizon discount factors.

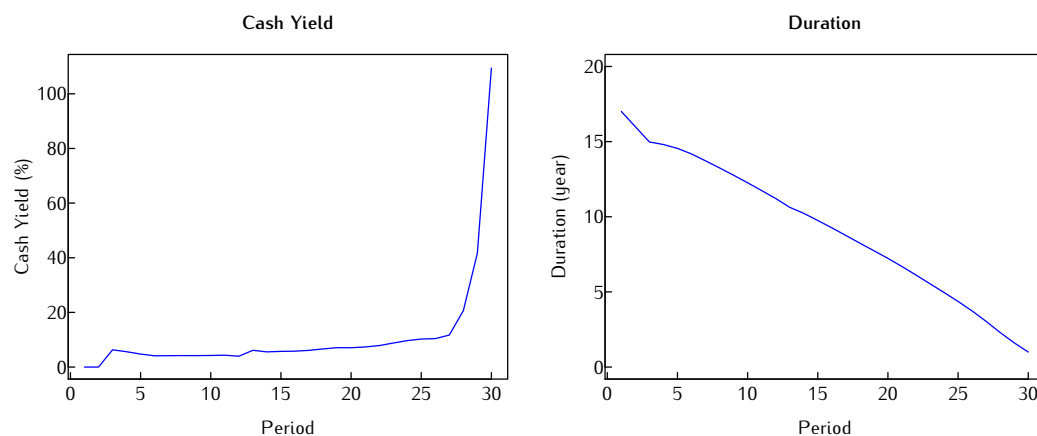
Conversely, the price is more uncertain in the beginning of the investment period

as it accumulates the variability in all the future discount factors, and becomes more certain as approach project maturity, since all investor prices converge at the project maturity.

Finally, figure 19 shows the cash yield (left panel), and the duration (right panel) of equity cash flows. Here, cash yield is defined as the ratio of each period's expected dividend to the previous period's

4. Illustration

Figure 19: Cash yield (left panel), defined as the ratio of the expected dividend to the lagged price, and the duration (right panel), defined as the sensitivity of the change in equity value to the changes in yield, of equity investment at each point in project's life. Increasing cash yield is a consequence of decreasing price due to fewer remaining dividends, and decreasing duration is decreasing time to maturity.



price (or lagged price). The expected cash yield increases in time reflecting that each period's dividend constitutes a bigger fraction of the previous period's price, as the number of remaining dividends decrease. It starts at zero, as the expected dividend during construction phase is zero, and goes above 100% in the final period as the expected dividend in the last period exceeds the price in the previous period, which is simply the discounted value of the last dividend.

Similarly, duration decreases in time, as the number of remaining cash flows decrease, decreasing the effective maturity of the equity investment.

4.3 Sensitivity to Conditional Dividend Volatility Estimates

In this section, we test the sensitivity our results to changes in the assumed conditional volatilities of future dividends.

This also tests the possibility of a review of volatility forecasts by market participants in the case, for example, of infrastructure investments that are exposed to the business cycle, e.g. toll roads.

We consider a change of forecasted average dividend volatility from an initial estimate of 10% to 110%. We then compute the implied prices and IRRs for these different levels of conditional volatilities.

Figure 20 shows the change in price (left panel) and IRR (right panel) with changing volatility forecasts. As expected, an increase in a project's riskiness decreases the price, and increases the IRR, as investors require a higher return to be compensated for a higher perceived risk.

4.4 Market Dynamics

Finally, we can use our model to track the level as well as the bounds of observed market prices and the implied IRRs

4. Illustration

Figure 20: Filtered prices and term structure with varying expectations of dividend volatility: The expectations regarding dividend volatility are assumed to vary from an average volatility of 10% to an average volatility of 110%. The corresponding filtered prices and IRRs are shown in the left and right panels, respectively.

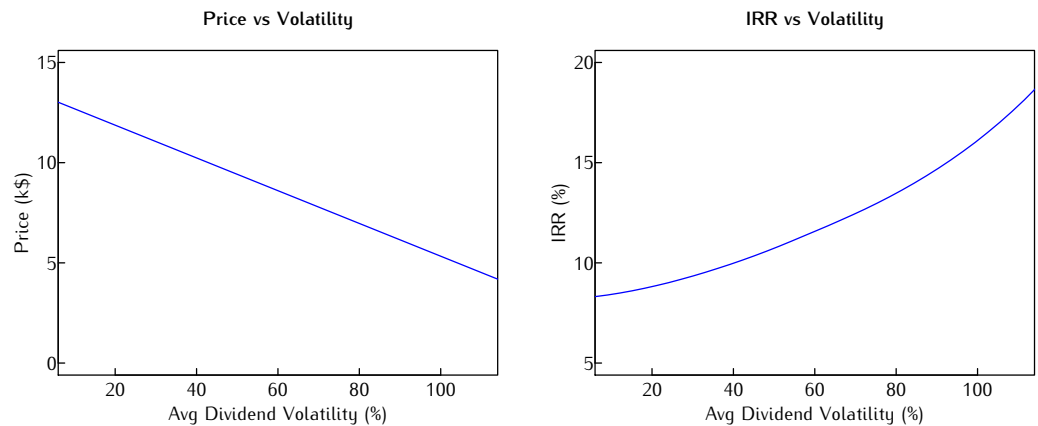
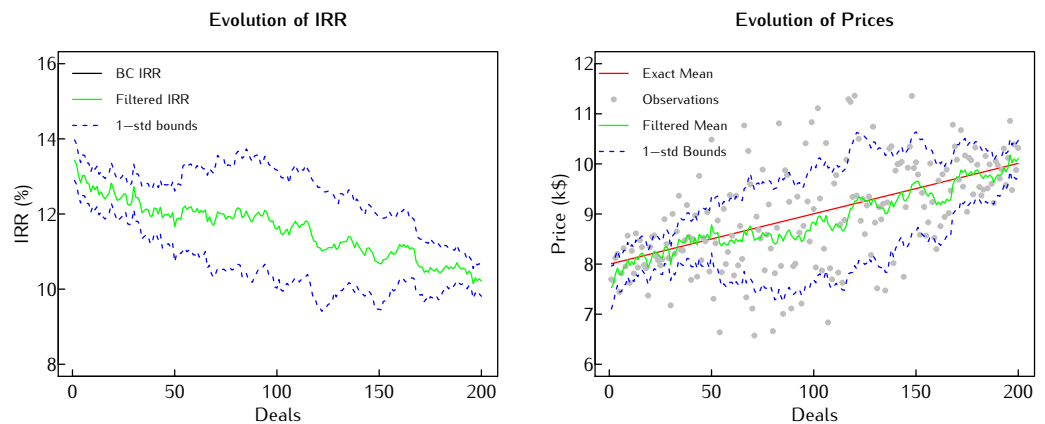


Figure 21: Filtered estimates of the IRR (left panel) and transaction prices (right panel), with one standard deviation bounds.



(computed using the correctly computed price).

To illustrate this, we assume observing a vector of 200 transaction prices. These prices are assumed to have an upward trend, and a time-varying standard deviation. In other words, in this example market, over the considered period, infrastructure equity investments tend to be more expensive (perhaps because investors value predictable cash flows more) but the range of prices

investors are willing to pay for this (given) risk is assumed to evolve in time.

In this case, it first increases and then decreases: the standard deviation of prices increases from 5% to 15% for the first 100 transactions, and then decreases back to 5% over the next 100 transactions.

The initial level of prices is set such that the implied IRR of the first deals starts around 13% and then follows price movements.

4. Illustration

Figure 21 shows the filtered prices and implied IRRs for the assumed input transaction prices, which are shown in grey dots in the right panel. The filtered prices (green line in the right panel) move upward tracking the mean (true) transaction price (red line in the right panel), and the one-standard deviation bounds on filtered prices (dotted blue lines in the right panel) widen until the hundredth transaction and then shrink. Similarly, filtered IRRs (the green line in the left panel) decrease due to increasing prices, and their bounds (blue lines in the left panel) follows a similar evolution.

4.4.1 Evolution in time

Finally, we briefly discuss the evolution of the term structure in time as dividends are realised. Here, we assume that the first three expected dividends are realised in the first three years, and expectations regarding future expected dividends and dividend volatilities do not change. That is, expected dividends and dividend volatilities for future years are the same as shown in figure 13.

Assuming that the dividend distribution does not change, we then estimate the filtered term structure for the remaining cash flows after three dividend payments, at time $t + 3$, using an assumed vector of prices/values. In practice, these prices may be obtained from the implied average term structure of new deals that close in that year. For instance, one could use the transaction prices of identical 30-year projects closed at time $t + 3$, and use their implied term structures to value the remaining 27 cash flows of existing projects.

Moreover, for demonstration purposes, we assume that equity values follow the upward trend shown in figure 21, and hence the valuation after three dividend periods are on average higher than the ones implied by the term structure implied by these investments when they were made.

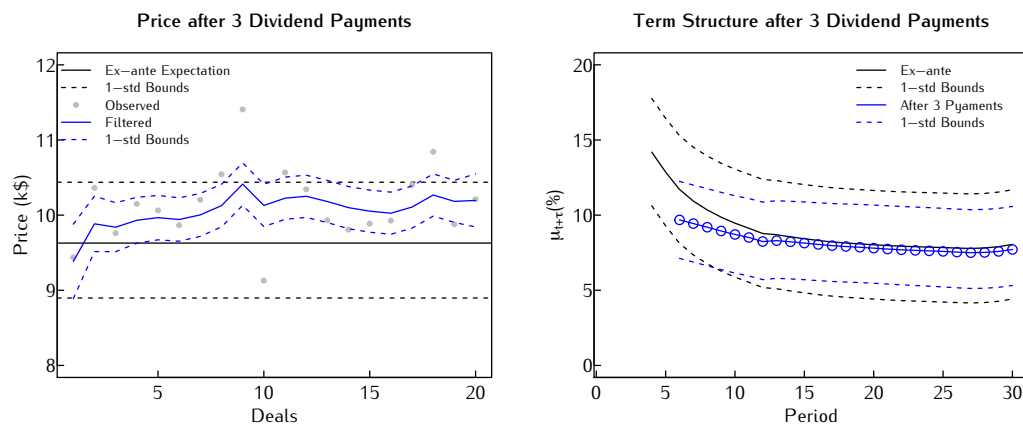
The *ex ante* expected price after three dividend payments, and the new valuations are shown in the left panel of the figure 22, and the implied term structure obtained after filtering over these observed transaction prices is shown in the right panel of the figure.

The new valuations (grey dots in the left panel) exceed the *ex ante* expected prices (black line in the left panel), and as a result the term structure (blue line in the right panel) shifts downward compared to the initial term structure (black line in the right panel). The near term discount rates show a larger downward shift as compared to the longer term discount rates, as near term rates are more sensitive to the observed prices.

Similarly, the bounds on filtered discount rates (dotted blue lines in the right panel) as well as the range of prices (dotted blue lines in the left panel) shrink compared to the *ex-ante* bounds (dotted black lines in the left and right panels), as the new valuations after three dividend payments have a smaller standard deviation (5%) compared to the initial transaction prices, which have an average standard deviation of 10%.

4. Illustration

Figure 22: Filtered prices and term structure after 3 dividend payments (at time $t_c + 3$): The assumed transaction prices, P_{t_c+3} , and the ex-ante expected price, $E_0 [P_{t_c+3}]$, are shown in the left panel. The ex-ante expected term structure, $\mu_{t_0+\tau}$ and the filtered term structure of expected returns after 3 dividend payments, $\mu_{t_c+3+\tau}$, are shown in the right panel. Realised prices exceed the ex-ante expectations and as a result, the term structure shifts downward.



5. Conclusions



5. Conclusions

In this paper, we propose a comprehensive framework to measure performance in privately-held infrastructure equity investments. In this context, we address two important challenges: the paucity of available data and the absence of a unique pricing measure for assets that are seldom traded, the value of which is determined by cash flows expected to occur several decades into the future. In particular, the absence of a unique price for comparable investments implies that investors' unobservable required rates of return on equity cannot be implied by a replicating portfolio.

We approach the problem in two steps: a model of expected cash flows and a valuation framework given the information available at the time of investment about the distribution of future dividends.

5.1 Cash Flow Model

In order to optimise the information available today about the distribution of dividends, we structure the cash flow model as a Bayesian inference problem, which yields **a list of required data to be collected** and can **incorporate what data exists today**.

The output of our cash flow model is the conditional distribution of the **equity service cover ratio** or ESCR at each point in the life of an infrastructure investment, a new metric, which we introduce in this paper.

In a given infrastructure project, $E(ESCR_t)$ captures the extent to which realised dividends meet the dividend stream defined in the project base case, which is known at the onset of the project and can be observed. In expectation, $ESCR_t$ combined with a base case dividend scenario is a direct measure of expected cash flows. Moreover, the variance of $ESCR_t$ is a direct measure of cash flow volatility. Thus, documenting the distribution of $ESCR_t$ for a given family of infrastructure equity investments, provides us with an input for our valuation framework.

We show that given a prior distribution of $ESCR_t$, we can obtain the meta-parameters of 1/ the probability to receive a dividend at time t and 2/ the distribution of $ESCR_t$ when a dividend is paid, and that once new empirical observations become available, posterior values can be derived that incorporate the most recent knowledge of the distribution of dividends in each infrastructure asset.

5.2 Valuation Framework

Once the conditional distribution of $ESCR_{t+\tau}$ is known, it can be used as the input of a state-space model linking observable transaction prices and expected cash flows to an unobservable term structure of discount factors.

As we expect infrastructure projects to change risk profile during their lifecycle, we use a formulation of the Gordon model as our observation equation, allowing for unique per period equity risk premia, the

5. Conclusions

product of which in each future period determines a term structure of discount factors.

We also show that this pricing equation can be written as a function of the expected value of $ESCR_t$ (or its log return) and base case dividends (or base case dividend growth rates). Our pricing equation nests the constant growth and discount rate Gordon model, and we show that it is a version of the two-period Gordon model with no terminal value i.e. in finite time, which is adequate for infrastructure equity investments with a pre-determined lifespan.

Next, we use a model of the term structure of required risk premia in each investment to derive a state equation. We assume that investors require autocorrelated returns from period to period but also adjust their return expectations as a function of the conditional volatility of cash flows.

We show that this state-space model can be formulated as a dynamic linear model (to an approximation) and be solved by applying a Kalman filter to derive the *term structure of risk premia implied by both expected cash flows, and observed initial investments*.

Moreover, dynamic linear modelling allows measuring an error term in the state equation, which we interpret as the reflection of the range of subjective investor valuation of the comparable asset. Thus, the required equity risk premium for each future period can be given a confidence interval, which reflects the bounded range

of expected returns for investments in the same underlying cash flow process.

5.3 Performance Measures

Given existing knowledge about the distribution of dividends in generic infrastructure projects and a series of observed equity investments and their base case, we can derive a forward curve of required risk premia and the associated term structure of discount factors of future dividends.

Later, once dividends are observed, realised returns can be computed, using realised states to update future cash flow expectations and compute an end of period value. Of course, realised returns remain conditional on current knowledge i.e. how well future cash flows are predicted at that stage for each type of infrastructure investment.

Other return measures can be computed using these results, including the implied costs of capital or equity IRR at each point in the life of the project and the returns obtained in excess of the original investment base case.

Finally, duration can also be computed, which provides highly relevant performance metrics to compare infrastructure investment with other the *liability-friendly* types of assets.

5. Conclusions

5.4 Next Steps: Data Collection

In this paper, we have accomplished the first three steps on the roadmap for the creation of long-term infrastructure equity investment benchmarks defined in Blanc-Brude (2014a).

Using a clear definition of the relevant financial instruments corresponding to privately-held infrastructure equity, e.g. a claim on future dividends with zero terminal value, we have devised a valuation methodology relying on modern asset pricing yet implementable given available data today, and we have determined a parsimonious set of data that can be collected to keep updating our valuation model and improve our knowledge of expected performance in privately-held infrastructure equity investments.

Next steps include the implementation of our data collection template to create a reporting standard for long-term investors and the ongoing collection of the said data.

In future research, we propose to develop models of return correlations for unlisted in infrastructure assets in order to work towards building portfolios of unlisted infrastructure assets.

6. Technical Appendix



6. Technical Appendix

6.1 Cash flow Metrics

6.1.1 Expected Cash Flows

Equity service cover ratio

For a stream of cash flow to equity C_t^i in each future state of the world i at time t , we can write,

$$ESCR_t^i = \frac{C_t^i}{C_t^0} \quad (6.1)$$

the **equity service cover ratio** (ESCR) at time t for state i , with $i = 0$, the base case and C_t^0 the base case dividend at time t . Hence, if realised dividend payments equal the base case, $ESCR_t = 1$.

Expected ESCR at time $t + \tau$

Each state i occurs with probability P_t^i and the expected value of $ESCR_t$ is written:

$$E_t(ESCR_{t+\tau}) = \sum_{i=1}^N \frac{P_{t+\tau}^i \times C_{t+\tau}^i}{C_{t+\tau}^0} \quad (6.2)$$

with $\tau = 1 \dots T$ remaining dividend periods and $i = 1 \dots N$ states

Standard deviation of the $ESCR_t$

The standard deviation of $ESCR_t$ follows the usual formulation:

$$\sigma ESCR_{t+\tau} = \sqrt{\sum_{i=1}^N P_{t+\tau}^i (ESCR_{t+\tau}^i - E_t(ESCR_{t+\tau}))^2} \quad (6.3)$$

$\sigma ESCR_{t+\tau}$ is a direct measure of conditional dividend volatility at time $t + \tau | t$.

6.1.2 Dividend Growth

ESCR return

The period change or return in ESCR between period t and $t + \tau$ is written:

$$\frac{ESCR_{t+\tau}}{ESCR_t}$$

and the log of ESCR return is

$$\begin{aligned} \text{escr}_{t+\tau} &= \ln\left(\frac{ESCR_{t+\tau}}{ESCR_t}\right) \\ &= \ln(ESCR_{t+\tau}) - \ln(ESCR_t) \end{aligned} \quad (6.4)$$

Finally, in expectation, we have:

$$\begin{aligned} E_t(\text{escr}_{t+\tau}) &= E_t(\ln(ESCR_{t+\tau}) - \ln(ESCR_t)) \\ &= E_t(\ln(ESCR_{t+\tau})) - E_t(\ln(ESCR_t)) \end{aligned}$$

with

$$E_t(\ln ESCR_t) \approx \ln(E_t(ESCR_t)) + \frac{\text{var}(ESCR_t)}{2 \times E(ESCR_t)^2}$$

Expected dividend growth rate at time t

$G_{t+\tau}^i$ the dividend growth rate in state i between time t and time $t + \tau$ is straightforwardly written:

$$G_{t+\tau}^i = \frac{C_{t+\tau}^i}{C_t^i} - 1 \quad (6.5)$$

and $g_{t+\tau}$ the log of the growth relative $1 + G_{t+\tau}^i$ is

$$\ln(1 + G_{t+\tau}^i) = \ln\left(\frac{C_{t+\tau}^i}{C_t^i}\right) \quad (6.6)$$

$G_{t+\tau}$ can be written as a function of the base case dividend growth rate at time $t + \tau$ and the ESCR return time t and $t + \tau$

$$\begin{aligned} G_{t+\tau} &= \frac{C_{t+\tau}^0 ESCR_{t+\tau}}{C_t^0 ESCR_t} - 1 \\ &= (1 + G_{t+\tau}^0) \frac{ESCR_{t+\tau}}{ESCR_t} - 1 \end{aligned}$$

Hence, $g_{t+\tau}$ the log of the growth relative $(1 + G_{t+\tau})$ is written

$$g_{t+\tau} = g_{t+\tau}^0 + \text{escr}_{t+\tau}$$

with $g_{t+\tau}^0$ the log of the base case dividend growth rates and

$$\begin{aligned} \text{escr}_{t+\tau} &= \ln\left(\frac{ESCR_{t+\tau}}{ESCR_t}\right) \\ &= \ln(ESCR_{t+\tau}) - \ln(ESCR_t) \end{aligned}$$

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In expectation at time t ,

$$\begin{aligned} E_t(G_{t+\tau}) &= \frac{C_{t+\tau}^0 E_t(ESCR_{t+\tau})}{C_t^0 E_t(ESCR_t)} - 1 \\ &= (1 + G_{t+\tau}^0) + \frac{E_t(ESCR_{t+\tau})}{ESCR_t} - 1 \end{aligned} \quad (6.7)$$

and using

$$\begin{aligned} E_t[g_{t+\tau}] &= E_t[\log(1 + G_{t+\tau})] \\ &\approx \log(E_t[1 + E_t[G_{t+\tau}]] + \frac{1}{2} \frac{\text{var}(G_{t+\tau})}{E_t[G_{t+\tau}]^2}) \\ &\approx \log(E_t[1 + E_t[G_{t+\tau}]]). \end{aligned}$$

Thus, using this approximation, the expected continuously compounded growth rate of ESCR can be written as

$$E_t(g_{t+\tau}) \approx g_{t+\tau}^0 + \text{escr}_{t+\tau}$$

6.1.3 Normalised equity service cover ratio

To control for construction cost overruns, we introduce the normalised equity service cover ratio or $nESCR$. While this risk has been documented to be low and well-managed in privately financed infrastructure projects (Blanc-Brude and Makovsek, 2014), we assume that (limited) construction cost overruns may have to be borne by the equity investor, and that additional equity capital is injected if initial investment costs increase.

To account for this, we can calculate a **normalised** $ESCR_t$ or $nESCR_t$, accounting for the variability of the initial investment.

$nESCR_t$ is written:

$$\begin{aligned} nESCR_t^i &= \frac{C_t^i}{C_0^i} / \frac{C_t^0}{C_0^0} \\ &= \frac{C_t^i}{C_t^0} / \frac{C_0^i}{C_0^0} \\ &= ESCR_t^i / \frac{C_0^i}{C_0^0} \end{aligned}$$

For C_0^0 the initial equity investments in the base case and C_0^i the initial equity investments in state i . If there is no construction risk, $nESCR_t = ESCR_t$. For simplicity we refer to $ESCR_t$ in the text and formulae. Empirically, we use $nESCR_t$ to control for the any increase in initial capital costs.

6.2 Risk Measures

6.2.1 Duration

Duration measures the sensitivity of a security's value to changes in the level of yield curve

$$\begin{aligned} D_t &= \frac{-1}{P_t} \frac{\partial P_t}{\partial y_t} \\ &= \frac{-1}{P_t} \frac{\partial}{\partial y_t} \sum_{i=t+1}^T e^{-y_t(i-t)} CF_i^{BC} \\ \Rightarrow D_t &= \frac{1}{P_t} \sum_{i=t+1}^T (i-t) e^{-y_t(i-t)} CF_i^{BC} \end{aligned}$$

where CF_i^{BC} is the base case equity payment at time i , P_t is the value of the equity at time t , and y_t is the yield at time t .

6.3 Return Measures

6.3.1 Yield to Maturity (IRR)

Yield to maturity, or equivalently IRR, is defined as the constant discount rate that

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makes the net present value of all the future cash inflows and outflows equal to zero. That is, it solves

$$P_t = \sum_{i=t+1}^T e^{-y_t(i-t)} CF_i$$

where CF_i is the cash flow at time i , P_t is the price at time t , and y_t is the yield (IRR) at time t .

6.3.2 Cash Yield

Cash yield is the ratio of dividend to lagged prices, and is often referred to as dividend yield, defined as

$$dy_t = \frac{CF_t}{P_{t-1}}$$

6.4 Bayesian Estimates

6.4.1 Estimating state transition probabilities

Each π_{ij} at time $t + i$ takes some value $\pi \in [0; 1]$ for each row of the matrix \mathbb{P}_{t+i} . Thus, each row of \mathbb{P}_{t+i} is equivalent to an independent Bernoulli draw of parameter π .

Say we can observe a population of N projects at time t , with n of successes (strictly positive dividend), this data (call it y) follows a binomial distribution with the likelihood:

$$\mathcal{L}(y|\pi) = \binom{N}{n} \pi^n (1 - \pi)^{N-n}$$

where $\binom{N}{n} = \frac{N!}{n!(N-n)!}$ is the binomial coefficient.

According to Bayes' Law:

$$p(\pi|y) \propto p(\pi)\mathcal{L}(y|\pi)$$

that is, the posterior (distribution) is proportional to the prior (distribution) times the likelihood.

We can give a *beta* prior density to $Pr(\pi)$, such that:

$$p(\pi; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \pi^{\alpha-1} (1 - \pi)^{\beta-1}$$

The Beta distribution has a domain on $[0,1]$ which can usefully represent a *probability* and can take *any* shape on its domain. The Beta distribution is also *conjugate* with respect to the Binomial likelihood, so that the product of the prior (Beta) and the likelihood (Binomial) is another Beta distribution, which incorporates the information obtained from observing the data.

$$\begin{aligned} p(\pi|y) &\propto p(\pi)\mathcal{L}(y|\pi) \\ &\propto \pi^n (1 - \pi)^{N-n} \times \pi^{\alpha-1} (1 - \pi)^{\beta-1} \\ &\propto \text{Beta}(\alpha + n, \beta + N - n) \end{aligned} \quad (6.8)$$

...to a normalising constant which does not depend on π .

Hence, the sufficient statistics to update the prior distribution of π are N and n , which we know to be observable.

Prior and posterior values of the meta-parameters α and β

For a given initial prior mean (μ_π) and variance (σ_π) of transition probability π , we compute the meta-parameters α and β thus

$$\begin{aligned} \alpha &= \left(\frac{(1 - \mu_\pi)}{\sigma_\pi} - \frac{1}{\mu_\pi} \right) \times \mu_\pi^2 \\ \beta &= \alpha \times \left(\frac{1}{\mu_\pi} - 1 \right) \end{aligned}$$

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Table 4: Posterior mean and variance of example transition probabilities at time t in each observation round

Round	$\mu_{\pi_{01}}$	$\sigma_{\pi_{01}}^2$	$\mu_{\pi_{11}}$	$\sigma_{\pi_{11}}^2$
0 (prior)	0.8	0.15	0.8	0.15
1	0.94990	0.00047	0.87995	0.00105
2	0.95495	0.00021	0.89497	0.00047
3	0.94663	0.00017	0.89998	0.00030
4	0.95247	0.00011	0.90498	0.00021
5	0.95398	0.00009	0.90399	0.00017
6	0.94998	0.00008	0.90166	0.00015
7	0.94284	0.00008	0.89571	0.00013
8	0.94499	0.00006	0.89749	0.00011
9	0.94332	0.00006	0.90333	0.00010
10	0.94799	0.00005	0.89899	0.00009
11	0.95181	0.00004	0.89818	0.00008
12	0.95082	0.00004	0.89416	0.00008
true values	0.95		0.9	

Table 5: Posterior mean and variance of example ESCR distribution parameters (logscale) at time t in each observation round

Round	μ_m	σ_m^2	μ_p	σ_p^2	$E(ESCR_t)$
0 (prior)	-0.1115	100	4.4814	50	1
1	-0.10918	0.019996	10.94110	2.43330	0.93850
2	-0.14124	0.00999	10.93431	1.20660	0.90891
3	-0.15899	0.006666	10.86998	0.79766	0.89315
4	-0.15464	0.00499	11.08961	0.61191	0.89623
5	-0.15203	0.003999	10.58602	0.44809	0.90051
6	-0.15245	0.003333	9.86847	0.32594	0.90323
7	-0.15317	0.002857	9.92916	0.28966	0.90230
8	-0.15424	0.002499	9.80691	0.24847	0.90190
9	-0.15898	0.002222	9.71768	0.20631	0.89805
10	-0.15654	0.00199	9.80043	0.19440	0.89985
11	-0.15432	0.00181	9.40519	0.16542	0.90379
12	-0.15733	0.00166	9.39092	0.15034	0.90115
true values	-0.1580		9.4912		0.9

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which follows from the definition of the beta density with mean $\frac{\alpha}{\alpha+\beta}$ and variance $\frac{\alpha\beta}{(\alpha+\beta^2)(\alpha+\beta+1)}$

Next, each time a new set of data (N, n) is observed, the posterior values of α and β can be computed according to equation 6.8 above. The posterior values of the mean and variance of π are then computed using the new values of α and β .

The resulting values for the example described in chapter 2 are given in table 4 for 12 iterations using 50 new observations in each round.

6.4.2 Estimating the meta-parameters of the conditional distribution of $ESCR_t$ in the payment state

If observed ESCR data (call it y) follow a lognormal process of mean m and precision p , then its likelihood function is given by (Fink, 1997):

$$\mathcal{L}(m, p|y) \propto p^{N/2} \exp\left(-\frac{p}{2} \sum_{n=1}^N (\ln(y) - m)^2\right)$$

The conjugate prior of a Lognormal process is a Gamma-Normal distribution, that is, as a function of m and p , this equation is proportional to the product of a Gamma function of p (with parameters a and b) with a Normal distribution (with mean μ and precision δ) of m conditional on p .

The sufficient statistics (required data) to update a prior distribution are the number of observations N , $\bar{X} = \frac{\sum_{n=1}^N \ln(X_n)}{N}$, and SS

the sum of squared deviation of the log data about m .

The marginal distribution of the precision parameter p follows the Gamma density function:

$$f(p; a, b) = \frac{p^{a-1} \exp(-\frac{p}{b})}{\Gamma(a) b^a}$$

And the marginal distribution of the mean parameter, m , is a t distribution with $2a$ degrees of freedom, location μ , and precision $a\delta b$:

$$f(m; a, b, \delta, \mu) = \sqrt{\frac{b\delta}{2\pi}} \frac{\Gamma(\frac{2a+1}{2})}{\Gamma(a)} \times \left(1 + \frac{\delta b}{2}(m - \mu)^2\right)^{-\frac{2a-1}{2}}$$

The joint posterior distribution $Pr(\hat{m}, \hat{p})$ is given by the meta-parameters (see Fink, 1997):

$$\begin{aligned} \hat{a} &= a + \frac{N}{2} \\ \hat{b} &= \left(\frac{1}{b} + \frac{SS}{2} + \frac{\delta N(\bar{X} - \mu)^2}{2(\delta + N)}\right)^{-1} \\ \hat{\mu} &= \frac{\delta\mu + N\bar{X}}{\delta + N} \\ \hat{\delta} &= \delta + N \end{aligned} \tag{6.9}$$

Prior and posterior values of the meta-parameters a, b, μ and δ

Next, for a given initial prior arithmetic mean ESCR ($E(ESCR_t)$) and arithmetic variance of ESCR at time t ($\sigma_{ESCR_t}^2$), we compute the prior parameters m (location)

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and *sigma* (scale) of the log data thus:

$$\begin{aligned}\sigma^2 &= \ln\left(1 + \frac{\sigma_{ESCR_t}^2}{E(ESCR_t)^2}\right) \\ m &= \ln(E(ESCR_t)) - \frac{1}{2}\ln\left(1 + \frac{\sigma_{ESCR_t}^2}{E(ESCR_t)^2}\right) \\ &= \ln(E(ESCR_t)) - \frac{1}{2}\sigma^2\end{aligned}$$

which follows from the definition of the lognormal density with mean $\exp(m + \frac{\sigma^2}{2})$ and variance $(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)$.

As discussed above, the value of parameter *m* follows a Gaussian (normal) distribution with meta-parameters μ (mean) and δ (precision). The prior value of μ is simply the prior value of *m* and the prior value of precision δ is set to a small number (implying a large variance), e.g. 0.01.

The value of precision parameter *p* follows a Gamma distribution for which we need to derive the shape (a) and rate (b) meta-parameters. We first give *p* a prior expected value and variance. The prior mean of *p* (call it μ_p) is simply the inverse of σ^2 , the initial prior for the scale of the log data. The prior variance of *p* (call it var_p) is set to a large number relative to the prior expected precision, e.g. ten times *p*.

The initial prior values of *a* and *b* are then computed as

$$\begin{aligned}a &= \frac{\mu_p^2}{var_p} \\ b &= \frac{\mu_p}{var_p}\end{aligned}$$

which follows from the definition of the Gamma density function with mean $\mu_p = ab$ and variance $var_p = ab^2$.

Thus, each time a new set of ESCR data is observed (*N*, \bar{X} and *SS*), the posterior values of \hat{a} , \hat{b} , $\hat{\mu}$ and $\hat{\delta}$ can be computed according to equation 6.9, and the posterior parameters \hat{m} and $\hat{\rho}$ of the distribution of $ESCR_t$ derived, incorporating prior knowledge and the new information.

The resulting values for the example described in chapter 2 are given in table 5 for 12 iterations using 50 new observations in each round.

6.5 Pricing equation

6.5.1 Pitfalls of using IRRs

Here, we present some examples of the aforementioned pitfalls of using IRR.

Robichek and Myers (1966) highlights a conceptual problem with using a single discount rate to adjust both for time value of money and riskiness of cash flows. Let the certainty equivalent of the expected cash flow at time *i*, $E_t[C_i]$, is C_i^* , i.e. C_i^* is the smallest certain amount that an investor would exchange for the risky amount C_i . Then the ratio $\alpha = \frac{C_i^*}{E_t[C_i]}$ captures preferences for risk. Now, assume that the greatest amount investor would pay now to receive the certain amount C_i^* at time *i* is $\frac{C_i^*}{(1+r)^i}$, then r^f captures the time value of money determined by the risk free rate, which is assumed to be constant in time. Thus, the correct present value of the uncertain cash flow C_i is $\frac{\alpha E_t[C_i]}{(1+r)^i}$. This exercise can be repeated for each cash flow of an investment that has multiple cash flows to

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obtain its value,

$$V_t = \sum_{i=1}^T \frac{E_t[C_i]}{(1+r^f)^i}. \quad (6.10)$$

Now we compare equation 6.10 to a valuation formula that does not distinguish between time and risk preferences, and instead uses a single constant discount rate to value these cash flows as

$$V_t = \sum_{i=1}^T \frac{\alpha_i E_t[C_i]}{(1+k)^i}, \quad (6.11)$$

where k can be interpreted as the IRR of the investment, as it is the constant rate that applies to all future cash flows.

In order for the values determined by equation 6.10 and equation 6.11 to be equal, we must have

$$\begin{aligned} \frac{\alpha_i E_t[C_i]}{(1+r^f)^i} &= \frac{E_t[C_i]}{(1+k)^i} \\ \Rightarrow \alpha_i &= \frac{(1+r^f)^i}{(1+k)^i}. \end{aligned}$$

This implies

$$\begin{aligned} \alpha_{i+1} &= \frac{(1+r^f)^{i+1}}{(1+k)^{i+1}} = \alpha_i \frac{(1+r^f)}{(1+k)}, \text{ and} \\ \frac{\alpha_{i+1} - \alpha_i}{\alpha_i} &= 1 - \frac{(1+r^f)}{(1+k)}, \end{aligned}$$

i.e. the ratio of certainty equivalent to expected cash flows decreases over time at a constant rate of $1 - \frac{(1+r^f)}{(1+k)}$. This implies that the investor would demand a lower certainty equivalent for the cash flows that lie farther in the future! This assumption, however, is unlikely to hold in general, and can produce erroneous valuations.

In order to see the effects of these two valuation equations, we consider two

investments that require the same initial investment I_t , but the first investment produces a single expected cash flow $E_t[C_{t+1}]$ at the end of the first year, and the second investment produces a single expected cash flow $E_t[C_{t+1}](1+r^f)$ at the end of the second year. If investor's preferences for risk are constant in time, and the investor requires the same certainty equivalent for a given expected cash flow i.e. $\alpha_{t+1} = \alpha_{t+2}$, then using equation 6.10

$$\begin{aligned} V(B) &= \frac{\alpha_{t+2} E_t[C_{t+1}](1+r^f)}{(1+r^f)^2} \\ &= \frac{\alpha_{t+2} E_t[C_{t+1}]}{(1+r^f)} \\ &= \frac{\alpha_{t+1} E_t[C_{t+1}]}{(1+r^f)} \\ &= V(A). \end{aligned}$$

However, using equation 6.11, we have

$$\begin{aligned} V(B) &= \frac{E_t[C_{t+1}](1+r^f)}{(1+k)^2} \\ &= \frac{E_t[C_{t+1}]}{(1+k)} \frac{(1+r^f)}{(1+k)} \\ &= V(A) \frac{(1+r^f)}{(1+k)}, \end{aligned}$$

which implies $V(B) < V(A)$ for any $k > r^f$. That is, if using equation 6.11 to compare the two investments, the investor would prefer the first project, which has a shorter duration, despite the fact that the second project compensates her opportunity cost for waiting, and the two investments are identical under the investor's true preferences. As mentioned before, this inconsistency simply arises because the valuation formula given in equation 6.11 makes a restrictive assumption about risk

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preferences ($\alpha_{t+2} < \alpha_{t+1}$), which is violated in our example as $\alpha_{t+2} = \alpha_{t+1}$.

Now, we look at the problems that can arise by using IRR to compute terminal value, duration, and expected losses of an investment. We consider two riskless investments A and B. Cash flows of the two investments are as follows. Investment A requires an initial investment of (\$100), returns \$400 in the next period, requires an additional investment of (\$400) in the third period, and returns \$100 in the fourth period. Investment B requires an initial investment of (\$100), requires an additional investment of (\$400) in the second period, returns \$400 in the third period, and returns \$100 in the fourth period. That is, overall cash flows of the two investments merely offset each other, and both investments differ only in the timing of the two intermediate cash flows of \$400.

Assuming a risk free rate of zero, the NPV of the two investments is zero. However, if we compute the IRR for the two investments, they look very different. While the IRR for investment B is zero, as expected, the IRR for investment A is a staggering 96%!¹⁶ This is due to the implicit reinvestment rate assumption of IRR, which assumes that all the cash flows can be re-invested at the IRR. Therefore, a high first cash flow of \$400 implies a high IRR, which combined with an assumption that it is re-invested at this high IRR leads to a completely mis-leading result. Indeed, if one could reinvest cash flows at 96%, this IRR would be economically justified as it would produce the correct terminal value

of the investment. If we reverse the order of the two intermediate \$400 cash flows (investment B), then the IRR becomes zero, as you do not make high return in any period so the re-investment assumption is harmless. This highlights the sensitivity of the IRR to the cash flow timings, and problem with multiple solutions to IRR-price equation.

Now, we look at the duration of these two investments computed using their IRRs. The investment A yields a duration of 0.53 years, while the investment B yields a duration of 7 years (which is also the duration under the risk-free rate), an error of 1200%!

Now, to see the effect of computing expected value of losses under IRR, consider that both investments are risky, and have an expected loss of 1%, and the annual constant risk-adjusted rate is 1%. That is, expected positive dividends for the two investments are \$396 and \$99. First, we recompute the IRRs for the two investments with these new expected cash flows. The IRR for investment A is now 0.93%, while the IRR for investment B is zero. Thus the expected present value of the losses is 0% for investment A and 5% for investment B. Computing losses under the risk adjusted rate of 1%, expected losses are 4.89%. Table 6 summarises these results.

6.5.2 Expected dividend growth rates

Traditional discounted cash flow models use dividend growth rates to capture the evolution of expected cash flows. We can write the cash flow at time $t + \tau$ as a function of g_t , the log of the

¹⁶ - This equation has two solutions, however, we choose the one just to highlight our point, as it produces the right present value, and one cannot discard it. In practice, which solution one arrives at depends where one starts the search. Moreover, it is easy to set the eliminate the $IRR = 0$ solution by changing the terminal cash flow.

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Table 6: This table compares the present value, terminal value, duration, and loss measures computed using their IRRs for two almost identical investments.

Time	Riskless Cash Flows		Expected Risky Cash Flows	
	Investment A	Investment B	Investment A'	Investment B'
0	-100	-100	-100	-100
1	400	-400	396	-400
2	-400	400	-400	396
3	100	100	99	99
IRR	96%	0	93%	-0.72%
Value	100	100	100	100
FV	1794	0	1633.48	97.87
Duration	0.53	7	0.53	7.04
E[Loss]	NA	NA	0	5%

dividend growth relative $(1 + G_t)$, defined in section 2.2.2.

$$C_{t+\tau} = C_t \prod_{j=1}^{\tau} \exp(g_{t+j}) = C_t \exp\left(\sum_{j=1}^{\tau} g_{t+j}\right)$$

That is, the cash flow at time $t + \tau$ is determined by the product of the cash flow at time t with the product of the exponential of the log rates of dividend growth (or the exponential of their sum) between $t + 1$ and $t + \tau$.

Next, $g_{t+\tau}$ can also be written as a function of the base case growth rate of dividends and $escr_{t+\tau}$, the log return of $ESCR_{t+\tau}$. If dividend growth is stochastic, the expected value of dividends is a function of the expected value of the ESCR return, since C_t and the base case are known at time t . It is written:

$$E_t(C_{t+\tau}) = C_t E_t e^{\sum_{j=1}^{\tau} (g_{t+j}^0 + escr_{t+j})}$$

Hence, the asset pricing formula is written:

$$\begin{aligned} P_t &= \sum_{\tau=1}^T E_t \left(e^{-\sum_{i=0}^{\tau-1} r_{t+i}} C_t e^{\sum_{j=1}^{\tau} (g_{t+j}^0 + escr_{t+j})} \right) \\ &= C_t \sum_{\tau=1}^T E_t \left(e^{\sum_{j=1}^{\tau} (g_{t+j}^0 + escr_{t+j} - r_{t+j-1})} \right) \end{aligned} \quad (6.12)$$

$$\begin{aligned} &= ESCR_t C_t^0 E_t \left[\sum_{\tau=1}^T e^{\sum_{j=1}^{\tau} (g_{t+j}^0 + escr_{t+j} - r_{t+j-1})} \right] \\ &= ESCR_t C_t^0 \sum_{\tau=1}^T E_t e^{\sum_{j=1}^{\tau} (g_{t+j}^0 + escr_{t+j} - r_{t+j-1})} \end{aligned} \quad (6.13)$$

since $ESCR_t$, C_t^0 , the base case dividend and g_{t+j}^0 the base case dividend growth rates, and conditional distributions of $escr_{t+j}$ and r_{t+i} are all known at time t .

The expression $\exp(\sum_{j=1}^{\tau} (g_{t+j}^0 + escr_{t+j}) - \sum_{i=0}^{\tau-1} r_{t+i})$ in equation (6.12) is a form of growth-adjusted discount factor, and yields the current asset value when applied iteratively to the current dividend payout over the remaining investment period T .

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We note that the pricing equation can be written solely as a function of base case dividends, base case dividend growth rates, and the distribution of ESCR log returns, all of which are known at time t , as well as the expected individual period discount rates, which are unknown.

We note that equation (6.12) nests the standard Gordon model (Gordon and Shapiro, 1956) with constant dividend growth and expected returns. In fact, this formulation can be considered a version of the Gordon model in finite time, with no terminal value, thus well-suited for dividend yielding securities with a finite life such as infrastructure project equity.

As shown in appendix 6.5.3, equation (6.12) can be written as the first term of a two-stage Gordon growth model with constant dividend growth and expected returns (Gordon and Gordon, 1997). In our setting, the second term of the two-stage Gordon model equals zero because dividend distributions stop after T and terminal value is zero.

6.5.3 The Pricing Equation as a Two-Stage Gordon Growth Model

Our model is a form of the Gordon model in finite time, with no terminal value. Equation 6.12 nests the standard Gordon model (Gordon and Shapiro, 1956) with constant dividend growth and constant expected returns. If dividend growth and expected returns are constant and static so that $g_t = \bar{g}$ and $r_t = \bar{r}$, equation 6.12 is

written:

$$P_t = C_t \left[\sum_{\tau=1}^T \exp(\tau\bar{g} - \tau\bar{r}) \right] \\ = C_t \left[\sum_{\tau=1}^T \exp(\tau(\bar{g} - \bar{r})) \right]$$

with $\bar{g} = \ln(1 + \bar{G})$ and $\bar{r} = \ln(1 + \bar{R})$,

$$P_t = C_t \left[\sum_{\tau=1}^T \exp((\ln(1 + \bar{G}) - \ln(1 + \bar{R}))\tau) \right] \\ = C_t \left[\sum_{\tau=1}^T \exp\left(\ln\left(\frac{1 + \bar{G}}{1 + \bar{R}}\right)^\tau\right) \right] \\ = C_t \left[\sum_{\tau=1}^T \left(\frac{1 + \bar{G}}{1 + \bar{R}}\right)^\tau \right] \\ = C_t \left[\sum_{\tau=1}^{\infty} \left(\frac{1 + \bar{G}}{1 + \bar{R}}\right)^\tau - \sum_{\tau=T+1}^{\infty} \left(\frac{1 + \bar{G}}{1 + \bar{R}}\right)^\tau \right] \\ = C_t \left[\sum_{\tau=1}^{\infty} (k)^\tau - \sum_{\tau=T+1}^{\infty} (k)^\tau \right]$$

where $k = \left(\frac{1 + \bar{G}}{1 + \bar{R}}\right)$

Since

$$\sum_{\tau=1}^{\infty} k^\tau = k \sum_{\tau=1}^{\infty} k^{\tau-1} = k \sum_{i=0}^{\infty} k^i = k \frac{1}{1 - k} \\ = \frac{1 + \bar{G}}{\bar{R} - \bar{G}}$$

then,

$$\sum_{\tau=T+1}^{\infty} k^\tau = k^{T+1} \sum_{\tau=T+1}^{\infty} k^{\tau-T-1} = k^{T+1} \sum_{i=0}^{\infty} k^i \\ = k^{T+1} \frac{1}{1 - k} = \frac{1 + \bar{G}}{\bar{R} - \bar{G}} \left[\frac{1 + \bar{G}}{1 + \bar{R}} \right]^T$$

and

$$P_t = C_t \frac{1 + \bar{G}}{\bar{R} - \bar{G}} \left[1 - \left(\frac{1 + \bar{G}}{1 + \bar{R}}\right)^T \right]$$

which is the first term of the two-stage Gordon growth model (Gordon and

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Gordon, 1997) with constant growth and discount rates. The second term of the two-stage Gordon growth model, given by $C_{T+1} \frac{1+G^\infty}{(R-G^\infty)(1+R)^T}$, equals zero when dividend distributions stop after T and terminal value is zero, which is assumed to be the case here.

6.5.4 Linearising the observation equation

From equation 3.2, we have:

$$P_t = E_t[C_{t+1}] e^{-r_t} + E_t[C_{t+2}] e^{-r_t} e^{-r_{t+1}} + \dots + E_t[C_{t+T}] e^{-r_t} e^{-r_{t+1}} \dots e^{-r_{t+T-1}} \quad (6.14)$$

$$P_t = (E_t[C_{t+1}] + \dots + E_t[C_{t+T}]) - (r_t^f + \lambda_t) (E_t[C_{t+1}] + \dots + E_t[C_{t+T}]) - \dots - (r_{t+T-1}^f + \lambda_{t+T-1}) [C_{t+T}]$$

$$= \begin{bmatrix} E_t[C_{t+1}] & \dots & E_t[C_{t+T}] \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} E_t[C_{t+1}] & \dots & E_t[C_{t+T}] \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \cdot$$

$$\left(\begin{bmatrix} r_t^f \\ \vdots \\ r_{t+T-1}^f \end{bmatrix} + \begin{bmatrix} \lambda_t \\ \vdots \\ \lambda_{t+T-1} \end{bmatrix} \right) = \underset{1 \times T}{D'} \cdot \underset{T \times 1}{1} - \underset{1 \times T}{D'} \cdot \underset{T \times T}{A} \cdot \underset{T \times 1}{(R_t^f + \Lambda_t)}, \quad (6.15)$$

where

$$\underset{T \times 1}{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \underset{T \times 1}{D} = \begin{bmatrix} E_t[C_{t+1}] \\ E_t[C_{t+2}] \\ \vdots \\ E_t[C_{t+T}] \end{bmatrix}$$

is the vector of expected dividends,

$$\underset{T \times T}{A} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

is a lower triangular matrix,

$$\underset{T \times 1}{R_t^f} = \begin{bmatrix} r_t^f \\ r_{t+1}^f \\ \vdots \\ r_{t+T-1}^f \end{bmatrix}$$

is a vector of risk-free rates at T horizons and

$$\underset{T \times 1}{\Lambda_t} = \begin{bmatrix} \lambda_t \\ \lambda_{t+1} \\ \vdots \\ \lambda_{t+T-1} \end{bmatrix}$$

is a vector of per-period excess returns at the same horizons.

Combining all the observable quantities (price, risk free rates, and expected cash flows) on the left hand side, we can re-write equation 6.15 as

$$\underset{T \times 1}{D'} \cdot \underset{T \times 1}{1} - \underset{1 \times T}{D'} \cdot \underset{T \times T}{A} \cdot \underset{T \times 1}{R_t^f} - P_t = \underset{1 \times T}{D'} \cdot \underset{T \times T}{A} \cdot \underset{T \times 1}{\Lambda_t} \quad (6.16)$$

Equation 6.16 provides a linear relationship between observed prices and the vector of period excess returns Λ_t .

6.5.5 Analytical derivation of the term structure

Starting at project maturity, we can write,

$$P_T = 0 \quad (6.17)$$

$$P_{T-1} = \frac{C_T + P_T}{1 + r_{T-1}^f + \lambda_{T-1}}, \quad (6.18)$$

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using equation 3.12

$$\begin{aligned}\lambda_{T-1} &= \frac{1}{\phi} [\lambda_T - \gamma_T \dot{\sigma}_{\lambda_T}] \\ &= \frac{1}{\phi} [\lambda_T - \gamma_T (\sigma_{\lambda_T} - \sigma_{\lambda_{T-1}})],\end{aligned}$$

since cash flows from time T (project maturity) onward are known to be zero with full certainty, λ_T and σ_T are both zero, and we can write

$$\lambda_{T-1} = \frac{\gamma_{T-1}}{\phi} \sigma_{\lambda_{T-1}}.$$

Equation 6.18 can then be written as

$$P_{T-1} \left[1 + r_{T-1}^f + \frac{\gamma_{T-1}}{\phi} \sigma_{\lambda_{T-1}} \right] = C_T$$

Since, $1 + r_{T-1}^f + \lambda_{T-1} = \frac{C_T}{P_{T-1}}$, if the risk free rate is deterministic, $\sigma_{\lambda_{T-1}} = \frac{\sigma_{C_T}}{P_{T-1}}$, as P_{T-1} is known at $T-1$. Thus

$$\begin{aligned}P_{T-1} \left[1 + r_{T-1}^f + \frac{\gamma_{T-1}}{\phi} \frac{\sigma_{C_T}}{P_{T-1}} \right] &= C_T \\ P_{T-1} [1 + r_{T-1}^f] &= C_T - \frac{\gamma_{T-1}}{\phi} \sigma_{C_T} \\ P_{T-1} &= \frac{C_T - \frac{\gamma_{T-1}}{\phi} \sigma_{C_T}}{1 + r_{T-1}^f}.\end{aligned}\quad (6.19)$$

Equation 6.19 simply states that the price at $T-1$ is the next period's cash flow adjusted by a risk premium, which is investor's risk aversion times the volatility of cash flow, discounted one period by the risk free rate. Similarly, the price at $T-2$ can be obtained as follows

$$\begin{aligned}P_{T-2} &= \frac{C_{T-1} + P_{T-1}}{1 + r_{T-2}^f + \lambda_{T-2}} \\ P_{T-2} &= \frac{C_{T-1} + P_{T-1}}{1 + r_{T-2}^f + \frac{\gamma_{T-2}}{\phi} \sigma_{\lambda_{T-2}}}\end{aligned}$$

As before

$$\begin{aligned}\sigma_{\lambda_{T-2}} &= \sigma \left[\frac{C_{T-1} + P_{T-1}}{P_{T-2}} \right] \\ &= \frac{1}{P_{T-2}} \sigma \left[C_{T-1} + \frac{C_T - \frac{\gamma_{T-1}}{\phi} \sigma_{C_T}}{1 + r_{T-1}^f} \right] \\ &= \frac{1}{P_{T-2}} \left[\sigma_{C_{T-1}} + \frac{\sigma_{C_T}}{1 + r_{T-1}^f} \right],\end{aligned}\quad (6.20)$$

where equation 6.20 uses the fact that $C_T|T-1$ is independent of $C_{T-1}|T-2$. Then P_{T-2} can be written as

$$\begin{aligned}P_{T-2} (1 + r_{T-2}^f + \frac{\gamma_{T-2}}{\phi} \sigma_{\lambda_{T-2}}) &= C_{T-1} + P_{T-1} \\ P_{T-2} \left[1 + r_{T-2}^f + \frac{\gamma_{T-2}}{\phi} \frac{1}{P_{T-2}} \left[\sigma_{C_{T-1}} + \frac{\sigma_{C_T}}{1 + r_{T-1}^f} \right] \right] & \\ &= C_{T-1} + \frac{C_T - \frac{\gamma_{T-1}}{\phi} \sigma_{C_T}}{1 + r_{T-1}^f} \\ P_{T-2} [1 + r_{T-2}^f] + \frac{\gamma_{T-2}}{\phi} \sigma_{C_{T-1}} + \frac{\gamma_{T-2}}{\phi} \frac{\sigma_{C_T}}{1 + r_{T-1}^f} & \\ &= C_{T-1} + \frac{C_T - \frac{\gamma_{T-1}}{\phi} \sigma_{C_T}}{1 + r_{T-1}^f},\end{aligned}$$

which can be written as

$$\begin{aligned}P_{T-2} &= \frac{C_{T-1}}{1 + r_{T-2}^f} + \frac{C_T}{(1 + r_{T-2}^f)(1 + r_{T-1}^f)} - \\ &\frac{\gamma_{T-2} \sigma_{C_{T-1}}}{\phi(1 + r_{T-2}^f)} - \frac{(\gamma_{T-2} + \gamma_{T-1}) \sigma_{C_T}}{\phi^2(1 + r_{T-2}^f)(1 + r_{T-1}^f)}\end{aligned}\quad (6.21)$$

Equation 6.21 implies that the price at $T-2$ is the present value of future cash flows discounted at the risk free rates, adjusted by the risks of future cash flows.

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Thus, prices for all times can be computed in this recursive manner

$$P_{t_c} = \sum_{\tau=1}^T \left[\frac{C_{t+\tau} - \frac{1}{\phi^\tau} (\sum_{i=1}^{\tau} \gamma_{t+i}) \sigma_{C_{t+\tau}}}{\prod_{j=0}^{\tau-1} (1 + r_{t+j}^f)} \right]$$

$$P_{t_0} = \exp\left(-\sum_{j=0}^{t_c-t_0-1} r_{t_0+j}^f\right) P_{t_c}, \tag{6.22}$$

where P_{t_c} is the price at construction completion (one period before the first dividend), and P_{t_0} is the price at the financial close of the project. This equation does not contain expected excess returns, λ_t , and return volatilities, σ_{λ_t} , as those have been expressed in terms of cash flow characteristics. This equation merely states that the price is the present value of the future cash flows adjusted for their volatilities, and discounted by the risk free rate. Thus, equation 6.22 can be interpreted as the analogue of equations 3.3 and 3.4 in the risk neutral framework, so that the numerator is the risk adjusted (risk neutral) expected cash flow, which is less than the expected cash flow by the required risk premium (volatility of cash flow times the price of risk), and the denominator is the risk free discount rate.

Using the prices determined as a function of cash flow characteristics and investor preferences in equation 6.22, prior estimate for the expected excess return at time t and their volatilities can be derived as follows

$$\lambda_t = \frac{P_{t+1} + C_{t+1}}{P_t} - r_t^f - 1 \tag{6.23}$$

$$\sigma_{\lambda_{t-1}} = \sigma_{\lambda_t} - \frac{\lambda_t - \phi \lambda_{t-1}}{\gamma_t} \tag{6.24}$$

starting with $\sigma_T = 0$.

6.5.6 Maximum likelihood estimation of the model parameters

Implementing MLE involves matching the distribution of model predicted prices and the selected observed prices. Given n random observations, y_1, \dots, y_n , we denote the joint density of the observations by $p(y_1, \dots, y_n; \psi)$, which depends on a vector of unknown parameters

$$\psi = \{Q_1, \dots, Q_n, R_1, \dots, R_n, \phi, \gamma_1, \dots, \gamma_T\} \tag{6.25}$$

where Q_1, \dots, Q_n are covariance matrices of the state transition equation, R_1, \dots, R_n are the variances of the observation equation, and ϕ and $\gamma_1, \dots, \gamma_T$ are the parameters of the discount rate model.

In a DLM, the price for k^{th} transaction is normally distributed with mean f_k and variance S_k . Thus, $p(y_1|y_{0:0}), p(y_2|y_{0:1}), \dots, p(y_k|y_{0:k-1})$ are independent, and the joint density, $p(y_1, \dots, y_n; \psi)$, can be written as a product of individual densities

$$p(y_1, \dots, y_n; \psi) = \prod_{k=1}^n p(y_k|y_{0:k-1}; \psi) \tag{6.26}$$

where $p(y_k|y_{0:k-1}; \psi)$ is the conditional density of y_k given the information up to $(k - 1)^{th}$ transaction. Each one of the densities appearing in the product in equation 6.26 is Gaussian with mean f_k and variance S_k . Therefore, the log-likelihood function can be written as

$$l(\psi) = -\frac{1}{2} \sum_{k=1}^n \log |S_k| - \frac{1}{2} \sum_{k=1}^n \frac{(y_k - f_k)^2}{S_k} \tag{6.27}$$

where f_k and S_k depend on ψ . The maximum likelihood estimate of model parameters, $\hat{\psi}$,

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can be obtained numerically as

$$\hat{\psi} = \max_{\psi} J(\psi)$$

Maximising equation 6.27 is roughly equivalent to minimising the sum of the prediction errors (distance between the average model predicted prices, f_k , and the observed prices), y_k , and the variance of the model predictions, S_k . In other words, the vector of unknown parameters, ψ , is roughly determined such that the distribution of predicted prices is as localised as possible (lower variance), while tracking observed prices as closely as possible (lower prediction error).

The vector of unknown parameters defined in equation 6.25, however, may contain more parameters than can be reliably estimated with limited data. To reduce the number of parameters, we can assume that the covariance matrix of state equation, Q_k , is a diagonal matrix. This amounts to assuming that the evolution of discount rates between deals is such that the $(t + \tau)^{th}$ period discount rate for $(k + 1)^{th}$ transaction, $r_{t+\tau}^{k+1}$, depends only on the $(t + \tau)^{th}$ period discount rate for k^{th} transaction, $r_{t+\tau}^k$. However, since the discount rates are serially correlated for k^{th} deal, i.e. $r_{t+\tau}^k \propto r_{t+\tau-1}^k$, the discount rates for the $(k + 1)^{th}$ deal are also serially correlated with each other, i.e. $r_{t+\tau}^{k+1} \propto r_{t+\tau-1}^{k+1}$. Thus, each discount rate for $(k + 1)^{th}$ deal, $r_{t+\tau}^{k+1}$, is correlated both with the previous period's discount rate for the same deal, $r_{t+\tau-1}^{k+1}$, as well as with the same period's discount rate of the previous deal, $r_{t+\tau}^k$. If we further assume that forecasting uncertainty is the

same for all discount rates, then

$$Q_k = \sigma_s^2(k) \frac{I}{T \times T}$$

where $\frac{I}{T \times T}$ is an identity matrix, and its estimation at time t reduces to estimating one number $\sigma_s^2(t)$, considerably simplifying the estimation.¹⁷

This MLE procedure can be repeated on a rolling basis using last n observations, y_{i-n}, \dots, y_i in equation 6.27 to update model parameters continuously across transactions. For example, if the distribution of input prices is time varying, this time variation can be captured in the distribution of foretasted prices by continuously estimating the Q and R parameters using the last n observations, y_{i-n}, \dots, y_i for every transaction, i , after the first n transactions (which are used in the first MLE). Thus, determining the unknown parameters on a rolling basis involves filtering over the last n observations for different choices of the parameters, and selecting the values that lead to filtered distributions that are most consistent with the observed prices.

6.5.7 Sensitivity of MLE estimates to changes in the initial conditions of the term structure parameters

Table 6.5.7 shows the effect of initial choice of $(\mathbf{y}_{t+i})_{i=1}^T$ values on filtered prices. Each column reports the results for one particular choice of $(\mathbf{y}_{t+i})_{i=1}^T$ and ϕ . Default choices are $(\mathbf{y}_{t+i})_{i=1}^T = 1$ and $\phi = 1$, and only the values shown in each column differ from its default value. That is, in the second column, $(\mathbf{y}_{t+i})_{i=1}^T = 0.1$, and $\phi = 1$, and so on. Then, for each column of the table, we

17 - Since the observation equation is one dimensional, R_t is already a one-dimensional number.

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Table 7: Comparison of filtered prices for different choices of initial values of $(\gamma_{t+i})_{i=1}^T$ and ϕ . All prices are in k\$. Filtered prices converge after about 10-12 iterations, irrespective of the choice of initial values of $(\gamma_{t+i})_{i=1}^T$ or ϕ .

n	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\phi = 0.1$	$\phi = 1$
1	6.03	7.43	6.99	10.38	6.35
2	6.13	6.8	6.59	8.23	6.28
3	6.3	6.72	6.59	7.62	6.39
4	6.41	6.7	6.61	7.32	6.47
5	6.3	6.51	6.45	6.96	6.35
6	6.31	6.47	6.42	6.8	6.35
7	6.42	6.54	6.5	6.78	6.45
8	6.43	6.52	6.49	6.7	6.45
9	6.56	6.62	6.6	6.76	6.57
10	6.46	6.51	6.5	6.62	6.48
11	6.52	6.56	6.55	6.64	6.53
12	6.48	6.5	6.5	6.57	6.48
13	6.6	6.62	6.61	6.67	6.6
14	6.56	6.57	6.57	6.61	6.56
15	6.63	6.65	6.64	6.67	6.64
16	6.66	6.67	6.67	6.69	6.67
17	6.62	6.62	6.62	6.64	6.62
18	6.57	6.57	6.57	6.59	6.57
19	6.43	6.43	6.43	6.44	6.43
20	6.51	6.47	6.49	6.44	6.52
21	6.61	6.57	6.6	6.47	6.62
22	6.69	6.66	6.69	6.53	6.7
23	6.67	6.65	6.67	6.54	6.68
24	6.64	6.63	6.64	6.54	6.64
25	6.59	6.58	6.59	6.52	6.6

start with a constant γ for all periods, and then compute the MLE estimates of these parameters by filtering over prices. As can be seen from the table, iterating over 10-12 prices almost completely eliminates the effects of initial choice on the filtered prices. That is, model forecasted prices using MLE estimates after iterating over 10-12 prices are almost entirely independent from initial choice of parameters. Similar results are obtained for different choices of ϕ values.

References



References

- Aguais, S., L. Forest, and D. Rosen (2000). Building a credit risk valuation framework for loan instruments. *Algo Research Quarterly* 3(3), 21–46.
- Ang, A., B. Chen, W. N. Goetzmann, and L. Phalippou (2013, November). Estimating Private Equity Returns from Limited Partner Cash Flows.
- Ang, A. and J. Liu (2004, December). How to Discount Cashflows with Time-Varying Expected Returns. *The Journal of Finance* 59(6), 2745–2783.
- Ben-Horim, M. and N. Sivakumar (1988, December). Evaluating capital investment projects. *Managerial and Decision Economics* 9(4), 263–268.
- Blanc-Brude, F. (2013). *Towards efficient benchmarks for infrastructure equity investments*. Meridiam, Campbell-Lutyens and EDHEC Research Chair on Infrastructure Equity Investment Management and Benchmarking. Singapore: EDHEC-Risk Institute.
- Blanc-Brude, F. (2014a). Benchmarking Long-Term Investment in Infrastructure. *EDHEC-Risk Institute Position Paper July*.
- Blanc-Brude, F. (2014b). Long-term investment in infrastructure. In *Yale SOM and EDHEC-Risk Institute Alternative Investment Seminars*, New Haven.
- Blanc-Brude, F., M. Hasan, and O. R. H. Ismail (2014). *Unlisted infrastructure debt performance measurement*, Volume July of *EDHEC and NATIXIS Research Chair on Infrastructure Debt Investment Solutions*. Singapore: EDHEC-Risk Institute.
- Blanc-Brude, F. and D. Makovsek (2014). How much construction risk do sponsors take in project finance?
- Brealey, R. A. and S. C. Myers (2014). *Principles of Corporate Finance* (eleventh ed.). McGraw Hill.
- Cappe, O., S. J. Godsill, and E. Moulines (2007). An overview of existing methods and recent advances in sequential Monte Carlo. *Proceedings of the IEEE* 95, 899–924.
- Carr, P., H. Geman, and D. B. Madan (2001a, October). Pricing and hedging in incomplete markets. *Journal of Financial Economics* 62(1), 131–167.
- Carr, P., H. Geman, and D. B. Madan (2001b, October). Pricing and hedging in incomplete markets. *Journal of Financial Economics* 62(1), 131–167.
- Cochrane, J. H. (2005, January). The risk and return of venture capital. *Journal of Financial Economics* 75(1), 3–52.

References

- Cooper, I. (2010). Consistent Valuation of Project Finance and LBO's Using the Flows-to-Equity Method.
- Driessen, J., T.-C. Lin, and L. Phalippou (2012, June). A New Method to Estimate Risk and Return of Nontraded Assets from Cash Flows: The Case of Private Equity Funds. *Journal of Financial and Quantitative Analysis* 47, 511–535.
- du Plessis, R. M. (1967). Poor Man's Explanantion of Kalman Filtering.
- Durbin, J. and S. J. Koopman (2012). *Time series analysis by state space methods* (second ed.). Oxford University Press.
- El Karoui, N. and M.-C. Quenez (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM journal on Control and Optimization* 33(1), 29–66.
- Elton, E. J., M. J. Gruber, S. J. Brown, and W. N. Goetzmann (2009). *Modern Portfolio Theory and Investment Analysis* (7 ed.). West Sussex: John Wiley & Sons.
- Esty, B. C. (1999). Improved techniques for valuing large-scale projects. *The Journal of Structured Finance*.
- Fink, D. (1997, December). A compendium of conjugate priors. *Montana The Magazine Of Western History*, 1–47.
- Froot, K. A. and J. C. Stein (1998, January). Risk management, capital budgeting, and capital structure policy for financial institutions: An integrated approach. *Journal of Financial Economics* 47(1), 55–82.
- Gordon, J. R. and M. J. Gordon (1997). The finite horizon expected return model. *Financial Analysts Journal*, 52–61.
- Gordon, M. J. and E. Shapiro (1956, October). Capital Equipment Analysis: The Required Rate of Profit. *Management Science* 3(1), 102–110.
- Graham, J. R. and C. R. Harvey (2001). The theory and practice of corporate finance: evidence from the field. *Journal of Financial Economics* 60(2), 187–243.
- Haley, C. W. (1984). Valuation and risk-adjusted discount rates. *Journal of Business Finance & Accounting* 11(3), 347–353.
- Hughes, J., J. Liu, and J. Liu (2009). On the relation between expected returns and implied cost of capital. In *Review of Accounting Studies*, Volume 14, pp. 246–259.

References

- Jenkinson, T., M. Sousa, and R. Stucke (2013). How Fair are the Valuations of Private Equity Funds? *SSRN Electronic Journal*.
- Kaplan, S. N. and R. S. Ruback (1995). The valuation of cash flow forecasts: An empirical analysis. *The Journal of Finance* 50, 1059–1093.
- Kaplan, S. N. and A. Schoar (2005). Private Equity Performance: Returns, Persistence, and Capital Flows. *The Journal of Finance* 60(4), 1791–1823.
- Korteweg, A. and M. Sorensen (2007, December). Estimating Risk and Return of Infrequently-Traded Assets: A Bayesian Selection Model of Venture Capital.
- Ljungqvist, A. and M. Richardson (2003). The cash flow, return and risk characteristics of private equity. *Working Paper*, 1–43.
- Petris, G., S. Petrone, and P. Campagnoli (2009). *Dynamic Linear Models with R*. Springer.
- Phalippou, L. (2008). The hazards of using IRR to measure performance: the case of private equity. *Available at SSRN 1111796*.
- Phalippou, L. (2013). Yale's Endowment Returns: Case Study in GIPS Interpretation Difficulties. *The Journal of Alternative Investments* 15(4), 97–103.
- Phalippou, L. and O. Gottschalg (2009). The performance of private equity funds. *Review of Financial Studies* 22(4), 1747–1776.
- Phalippou, L. and M. Zollo (2005). What drives private equity fund performance. *Unpublished working paper*.
- Robichek, A. A. and S. C. Myers (1966). Conceptual Problems in the Use of Risk-Adjusted Discount Rates. *The Journal of Finance* 21(4), 727–730.
- Samuelson, P. A. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review* 2, 41–49.
- Shearer, A. T. and L. R. Forest Jr (1997). Improving quantification of risk-adjusted performance within financial institutions. *Com. Lending Rev.* 13, 48–57.
- Woodward, S. E. (2004, August). Measuring Risk and Performance for Private Equity.

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About EDHEC-Risk Institute



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Founded in 1906, EDHEC is one of the foremost international business schools. Accredited by the three main international academic organisations, EQUIS, AACSB, and Association of MBAs, EDHEC has for a number of years been pursuing a strategy of international excellence that led it to set up EDHEC-Risk Institute in 2001. This institute now boasts a team of over 95 permanent professors, engineers and support staff, as well as 48 research associates from the financial industry and affiliate professors.

The Choice of Asset Allocation and Risk Management

EDHEC-Risk structures all of its research work around asset allocation and risk management. This strategic choice is applied to all of the Institute's research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset-liability management approaches; or orienting the concept of dynamic "core-satellite" investment management in the framework of absolute return or target-date funds.

Academic Excellence and Industry Relevance

In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the Institute's advisory board. This board is made up of internationally recognised researchers, the Institute's business partners, and representatives of major international institutional investors. Management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been conducted by the centre to date:

- Asset allocation and alternative diversification
- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the EDHEC-Risk locations in Singapore, which was established at the invitation of the Monetary Authority of Singapore (MAS); the City of London in the United Kingdom; Nice and Paris in France; and New York in the United States.

EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs or major research projects:

- **Core-Satellite and ETF Investment, in partnership with Amundi ETF**
- **Regulation and Institutional Investment, in partnership with AXA Investment Managers**
- **Asset-Liability Management and Institutional Investment Management, in partnership with BNP Paribas Investment Partners**
- **Risk and Regulation in the European Fund Management Industry, in partnership with CACEIS**
- **Exploring the Commodity Futures Risk Premium: Implications for Asset Allocation and Regulation, in partnership with CME Group**
- **Asset-Liability Management in Private Wealth Management, in partnership with Coutts & Co.**

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- **Asset-Liability Management Techniques for Sovereign Wealth Fund Management, in partnership with Deutsche Bank**
- **The Benefits of Volatility Derivatives in Equity Portfolio Management, in partnership with Eurex**
- **Structured Products and Derivative Instruments, sponsored by the French Banking Federation (FBF)**
- **Optimising Bond Portfolios, in partnership with the French Central Bank (BDF Gestion)**
- **Asset Allocation Solutions, in partnership with Lyxor Asset Management**
- **Infrastructure Equity Investment Management and Benchmarking, in partnership with Meridiam and Campbell Lutyens**
- **Investment and Governance Characteristics of Infrastructure Debt Investments, in partnership with Natixis**
- **Advanced Modelling for Alternative Investments, in partnership with Newedge Prime Brokerage**
- **Advanced Investment Solutions for Liability Hedging for Inflation Risk, in partnership with Ontario Teachers' Pension Plan**
- **The Case for Inflation-Linked Corporate Bonds: Issuers' and Investors' Perspectives, in partnership with Rothschild & Cie**
- **Solvency II, in partnership with Russell Investments**
- **Structured Equity Investment Strategies for Long-Term Asian Investors, in partnership with Société Générale Corporate & Investment Banking**

The philosophy of the Institute is to validate its work by publication in international academic journals, as well as to make it available to the sector through its position papers, published studies, and conferences.

Each year, EDHEC-Risk organises three conferences for professionals in order to present the results of its research, one in London (EDHEC-Risk Days Europe), one in Singapore (EDHEC-Risk Days Asia), and one in New York (EDHEC-Risk Days North America) attracting more than 2,500 professional delegates.

EDHEC also provides professionals with access to its website, www.edhec-risk.com, which is entirely devoted to international asset management research. The website, which has more than 58,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC's analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 1.5 million readers.

EDHEC-Risk Institute: Key Figures, 2011-2012

Nbr of permanent staff	90
Nbr of research associates	20
Nbr of affiliate professors	28
Overall budget	€13,000,000
External financing	€5,250,000
Nbr of conference delegates	1,860
Nbr of participants at research seminars	640
Nbr of participants at EDHEC-Risk Institute Executive Education seminars	182

About EDHEC-Risk Institute

The EDHEC-Risk Institute PhD in Finance

The EDHEC-Risk Institute PhD in Finance is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world's best universities, such as Princeton, Wharton, Oxford, Chicago and CalTech, and enjoying the support of the research centre with the greatest impact on the financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

Research for Business

The Institute's activities have also given rise to executive education and research service offshoots. EDHEC-Risk's executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes. In partnership with CFA Institute, it has developed advanced seminars based on its research which are available to CFA charterholders and have been taking place since 2008 in New York, Singapore and London.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements with the Operations Research and Financial Engineering department of Princeton University to set up a joint

research programme in the area of risk and investment management, and with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of investment management.

As part of its policy of transferring know-how to the industry, EDHEC-Risk Institute has also set up ERI Scientific Beta. ERI Scientific Beta is an original initiative which aims to favour the adoption of the latest advances in smart beta design and implementation by the whole investment industry. Its academic origin provides the foundation for its strategy: offer, in the best economic conditions possible, the smart beta solutions that are most proven scientifically with full transparency in both the methods and the associated risks.

EDHEC-Risk Institute Publications and Position Papers (2011-2015)



EDHEC–Risk Institute Publications (2011–2015)

2015

- Blanc-Brude, F., and M. Hasan. The Valuation of Privately-Held Infrastructure Equity Investments (January).
- Chhabra, A., R. Deguest, L. Martellini, V. Milhau and A. Suri. Introducing a Comprehensive Risk Allocation Framework for Goals-Based Wealth Management (January).

2014

- Coqueret, G., R. Deguest, L. Martellini, and V. Milhau. Equity Portfolios with Improved Liability-Hedging Benefits (December).
- Blanc-Brude, F., and D. Makovsek. How Much Construction Risk do Sponsors take in Project Finance. (August).
- Loh, L., and S. Stoyanov. The Impact of Risk Controls and Strategy-Specific Risk Diversification on Extreme Risk (August).
- Blanc-Brude, F. and F. Ducoulombier. Superannuation v2.0 (July)
- Loh, L., and S. Stoyanov. Tail Risk of Smart Beta Portfolios: An Extreme Value Theory Approach (July).
- Foulquier, P. M. Arouri and A. Le Maistre. P. A Proposal for an Interest Rate Dampener for Solvency II to Manage Pro-Cyclical Effects and Improve Asset-Liability Management (June).
- Amenc, N., R. Deguest, F. Goltz, A. Lodh, L. Martellini and E.Schirbini. Risk Allocation, Factor Investing and Smart Beta: Reconciling Innovations in Equity Portfolio Construction (June).
- Martellini, L., V. Milhau and A. Tarelli. Towards Conditional Risk Parity – Improving Risk Budgeting Techniques in Changing Economic Environments (April).
- Amenc, N., and F. Ducoulombier. Index Transparency – A Survey of European Investors Perceptions, Needs and Expectations (March).
- Ducoulombier, F., F. Goltz, V. Le Sourd, and A. Lodh. The EDHEC European ETF Survey 2013 (March).
- Badaoui, S., Deguest, R., L. Martellini and V. Milhau. Dynamic Liability-Driven Investing Strategies: The Emergence of a New Investment Paradigm for Pension Funds? (February).
- Deguest, R., and L. Martellini. Improved Risk Reporting with Factor-Based Diversification Measures (February).

EDHEC-Risk Institute Publications (2011-2015)

- Loh, L., and S. Stoyanov. Tail Risk of Equity Market Indices: An Extreme Value Theory Approach (February).

2013

- Lixia, L., and S. Stoyanov. Tail Risk of Asian Markets: An Extreme Value Theory Approach (August).
- Goltz, F., L. Martellini, and S. Stoyanov. Analysing statistical robustness of cross-sectional volatility. (August).
- Lixia, L., L. Martellini, and S. Stoyanov. The local volatility factor for asian stock markets. (August).
- Martellini, L., and V. Milhau. Analysing and decomposing the sources of added-value of corporate bonds within institutional investors' portfolios (August).
- Deguest, R., L. Martellini, and A. Meucci. Risk parity and beyond - From asset allocation to risk allocation decisions (June).
- Blanc-Brude, F., Cocquemas, F., Georgieva, A. Investment Solutions for East Asia's Pension Savings - Financing lifecycle deficits today and tomorrow (May)
- Blanc-Brude, F. and O.R.H. Ismail. Who is afraid of construction risk? (March)
- Lixia, L., L. Martellini, and S. Stoyanov. The relevance of country- and sector-specific model-free volatility indicators (March).
- Calamia, A., L. Deville, and F. Riva. Liquidity in european equity ETFs: What really matters? (March).
- Deguest, R., L. Martellini, and V. Milhau. The benefits of sovereign, municipal and corporate inflation-linked bonds in long-term investment decisions (February).
- Deguest, R., L. Martellini, and V. Milhau. Hedging versus insurance: Long-horizon investing with short-term constraints (February).
- Amenc, N., F. Goltz, N. Gonzalez, N. Shah, E. Shirbini and N. Tessaromatis. The EDHEC european ETF survey 2012 (February).
- Padmanaban, N., M. Mukai, L. Tang, and V. Le Sourd. Assessing the quality of asian stock market indices (February).
- Goltz, F., V. Le Sourd, M. Mukai, and F. Rachidy. Reactions to "A review of corporate bond indices: Construction principles, return heterogeneity, and fluctuations in risk exposures" (January).

EDHEC-Risk Institute Publications (2011–2015)

- Joenväärä, J., and R. Kosowski. An analysis of the convergence between mainstream and alternative asset management (January).
- Cocquemas, F. Towards better consideration of pension liabilities in european union countries (January).
- Blanc-Brude, F. Towards efficient benchmarks for infrastructure equity investments (January).

2012

- Arias, L, P. Foulquier and A. Le Maistre. Les impacts de Solvabilité II sur la gestion obligataire (December).
- Arias, L, P. Foulquier and A. Le Maistre. The Impact of Solvency II on Bond Management (December).
- Amenc, N., and F. Ducoulombier. Proposals for better management of non-financial risks within the european fund management industry (December).
- Cocquemas, F. Improving risk management in DC and hybrid pension plans (November).
- Amenc, N., F. Cocquemas, L. Martellini, and S. Sender. Response to the european commission white paper "An agenda for adequate, safe and sustainable pensions" (October).
- La gestion indicielle dans l'immobilier et l'indice EDHEC IEIF Immobilier d'Entreprise France (September).
- Real estate indexing and the EDHEC IEIF commercial property (France) index (September).
- Goltz, F., S. Stoyanov. The risks of volatility ETNs: A recent incident and underlying issues (September).
- Almeida, C., and R. Garcia. Robust assessment of hedge fund performance through nonparametric discounting (June).
- Amenc, N., F. Goltz, V. Milhau, and M. Mukai. Reactions to the EDHEC study "Optimal design of corporate market debt programmes in the presence of interest-rate and inflation risks" (May).
- Goltz, F., L. Martellini, and S. Stoyanov. EDHEC-Risk equity volatility index: Methodology (May).

EDHEC-Risk Institute Publications (2011-2015)

- Amenc, N., F. Goltz, M. Masayoshi, P. Narasimhan, and L. Tang. EDHEC-Risk Asian index survey 2011 (May).
- Guobuzaitė, R., and L. Martellini. The benefits of volatility derivatives in equity portfolio management (April).
- Amenc, N., F. Goltz, L. Tang, and V. Vaidyanathan. EDHEC-Risk North American index survey 2011 (March).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, L. Martellini, and S. Sender. Introducing the EDHEC-Risk Solvency II Benchmarks – maximising the benefits of equity investments for insurance companies facing Solvency II constraints - Summary - (March).
- Schoeffler, P. Optimal market estimates of French office property performance (March).
- Le Sourd, V. Performance of socially responsible investment funds against an efficient SRI Index: The impact of benchmark choice when evaluating active managers – an update (March).
- Martellini, L., V. Milhau, and A. Tarelli. Dynamic investment strategies for corporate pension funds in the presence of sponsor risk (March).
- Goltz, F., and L. Tang. The EDHEC European ETF survey 2011 (March).
- Sender, S. Shifting towards hybrid pension systems: A European perspective (March).
- Blanc-Brude, F. Pension fund investment in social infrastructure (February).
- Ducoulombier, F., L. Lixia, and S. Stoyanov. What asset-liability management strategy for sovereign wealth funds? (February).
- Amenc, N., F. Cocquemas, and S. Sender. Shedding light on non-financial risks – a European survey (January).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, L. Martellini, and S. Sender. Ground rules for the EDHEC-Risk Solvency II Benchmarks (January).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, L. Martellini, and S. Sender. Introducing the EDHEC-Risk Solvency II Benchmarks – maximising the benefits of equity investments for insurance companies facing Solvency II constraints - Synthesis - (January).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, L. Martellini, and S. Sender. Introducing the EDHEC-Risk Solvency II Benchmarks – maximising the benefits of equity investments for insurance companies facing Solvency II constraints (January).
- Schoeffler, P. Les estimateurs de marché optimaux de la performance de l'immobilier de bureaux en France (January).

EDHEC-Risk Institute Publications (2011–2015)

2011

- Amenc, N., F. Goltz, L. Martellini, and D. Sahoo. A long horizon perspective on the cross-sectional risk-return relationship in equity markets (December 2011).
- Amenc, N., F. Goltz, and L. Tang. EDHEC-Risk European index survey 2011 (October).
- Deguest, R., L. Martellini, and V. Milhau. Life-cycle investing in private wealth management (October).
- Amenc, N., F. Goltz, L. Martellini, and L. Tang. Improved beta? A comparison of index-weighting schemes (September).
- Le Sourd, V. Performance of socially responsible investment funds against an efficient SRI index: The impact of benchmark choice when evaluating active managers (September).
- Charbit, E., Giraud J. R., Goltz. F. and LTang. Capturing the market, value, or momentum premium with downside risk control: Dynamic allocation strategies with exchange-traded funds (July).
- Scherer, B. An integrated approach to sovereign wealth risk management (June).
- Campani, C.H. and F. Goltz. A review of corporate bond indices: Construction principles, return heterogeneity, and fluctuations in risk exposures (June).
- Martellini, L., and V. Milhau. Capital structure choices, pension fund allocation decisions, and the rational pricing of liability streams (June).
- Amenc, N., F. Goltz, and S. Stoyanov. A post-crisis perspective on diversification for risk management (May).
- Amenc, N., F. Goltz, L. Martellini, and L. Tang. Improved beta? A comparison of index-weighting schemes (April).
- Amenc, N., F. Goltz, L. Martellini, and D. Sahoo. Is there a risk/return tradeoff across stocks? An answer from a long-horizon perspective (April).
- Sender, S. The elephant in the room: Accounting and sponsor risks in corporate pension plans (March).
- Martellini, L., and V. Milhau. Optimal design of corporate market debt programmes in the presence of interest-rate and inflation risks (February).

EDHEC-Risk Institute Position Papers (2011-2015)

2014

- Blanc-Brude, F., Benchmarking Long-Term Investment in Infrastructure (July).

2013

- Amenc, N., F. Goltz and L. Martellini, Smart Beta 2.0 (June).

2012

- Till, H. Who sank the boat? (June).
- Uppal, R. Financial regulation (April).
- Amenc, N., F. Ducoulombier, F. Goltz, and L. Tang. What are the risks of European ETFs? (January).

2011

- Amenc, N., and S. Sender. Response to ESMA consultation paper to implementing measures for the AIFMD (September).
- Uppal, R. A short note on the Tobin Tax: The costs and benefits of a Tax on financial transactions (July).
- Till, H. A review of the G20 meeting on agriculture: Addressing price volatility in the food markets (July).

For more information, please contact:
Carolyn Essid on +33 493 187 824
or by e-mail to: carolyn.essid@edhec-risk.com

EDHEC–Risk Institute

393 promenade des Anglais BP 3116
06202 Nice Cedex 3 – France
Tel: +33 (0)4 93 18 78 24

EDHEC Risk Institute–Europe

10 Fleet Place - Ludgate
London EC4M 7RB - United Kingdom
Tel: +44 207 871 6740

EDHEC Risk Institute–Asia

One George Street - #07-02
Singapore 049145
Tel.: +65 6438 0030

EDHEC Risk Institute–North America

One Boston Place, 201 Washington Street
Suite 2608/2640, Boston, MA 02108
United States of America
Tel: +1 857 239 8891

EDHEC Risk Institute–France

16-18 rue du 4 septembre
75002 Paris
France
Tel: +33 (0)1 53 32 76 30